God, King, and Geometry: Revisiting the Introduction to Cauchy's *Cours d'Analyse*

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Abstract

This article offers a systematic reading of the introduction to Augustin-Louis Cauchy's landmark 1821 mathematical textbook, the *Cours d'analyse*. Despite its emblematic status in the history of mathematical analysis and, indeed, of modern mathematics as a whole, Cauchy's introduction has been more a source for suggestive quotations than an object of study in its own right. Cauchy's short mathematical metatext offers a rich snapshot of a scholarly paradigm in transition. A close reading of Cauchy's writing reveals the complex modalities of the author's epistemic positioning, particularly with respect to the geometric study of quantities in space, as he struggles to refound the discipline on which he has staked his young career.

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1. Introduction

Despite its emblematic status in the history of mathematical analysis and, indeed, of modern mathematics as a whole, the introduction to Augustin-Louis Cauchy's 1821 *Cours d'analyse* has been, in the historical literature, more a source for suggestive quotations than an object of study in its own right. In his introduction, Cauchy definitively outlines what were to be the foundations of his new rigorous mathematics, invoking both specific mathematical practices and their underlying philosophical principles. His text is thus a fecund encapsulation of the mathematical and epistemological work which would make him "the man who taught rigorous analysis to all of Europe" (Grabiner, 1981, p. 14).[1]

But Cauchy's short introduction is far more than a mere synopsis. A closer reading reveals it to be an active site of rhetorical positioning. This is especially the case if one follows Bottazzini's (2001, p. 34) suggestion to read the Course not as a textbook but as a manifesto. To appreciate better the significance of Cauchy's introduction, this article

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[1] For the work's spread and influence see also Belhoste (1991), Grattan-Guinness (1980, p. 97), and Grattan-Guinness (1990, pp. 795–796).

puts together three sources of evidence that have not been systematically combined hitherto. First, it offers a consideration of the introduction *as a whole*, detailing the relationships between the great variety of claims made therein. Second, it considers these relationships in the context of the extant detailed archival work on the influences and allusions in Cauchy's writing.[2] Finally, it begins the work of incorporating this analysis into broader historical frameworks.[3]

Read in this new light, Cauchy's introduction becomes a new sort of emblem—not of the dawn of modern analytic rigor, but of the specific rhetorical and philosophical negotiation such a revolution entails. It spans a mere seven full pages of the more than 1,500 pages of published material coming directly from Cauchy's teaching at the École Royale Polytechnique,[4] but in this short space it attempts radically to rework the ground of an entire discipline. As a brief and prefatory text, most of the work it does is through a deftly orchestrated web of allusions—to people, practices, and philosophical principles.[5] Choices that appear innocuous to a present-day eye, such as his allusion to Maclaurin's theorem on p. vi, become part of a dramatic staging of principle when viewed through the lens of contemporary disputes and discussions.

Cauchy's prefatory drama is set in three acts, corresponding to the three long paragraphs composing the bulk of the introduction. Each act enunciates the proper place of its subject within that subject's particular context. Thus, at the start, Cauchy positions his Course in Analysis within the engineering curriculum and the educational and civil missions of the École. In the second act, the one that has drawn the most attention from historians of mathematics to date, Cauchy elaborates the proper position of algebra and its associated methods within mathematics as a whole. The third act entertains a consideration of the proper place of mathematics in the spectrum of knowledge and within a broader society. The introduction concludes with a brief epilogue supplementing the expressions of gratitude with which Cauchy began.

This article treats these three acts in their order of appearance, arguing that Cauchy's rhetorical achievement was to stage his teaching, methods, and discipline in such a way as to valorize a very particular program of geometrical rigor. A synthetic reading of the introduction shows to a new degree the extent to which Cauchy's rigor was founded on an edifice of interconnected assertions spanning every level of knowledge, from the local to the universal.

2. Staging the Course

Founded in 1795, the École Polytechnique was from its inception France's premier training ground for the military, social, and civil engineers who were to be vital to the post-Revolutionary French state. Gilain (1989) and Belhoste (1991) have meticulously documented the mathematical debates that shaped Cauchy's education at the École,

[2] Particularly, Belhoste (1984, 1985, 1991); Gilain (1989).

[3] E.g. Richards (1991, 1992, 2006).

[4] The tabulation is from Gilain (1989, p. 24).

[5] Introductions to mathematical or scientific textbooks can be seen to constitute a genre in their own right, with a distinctive combination of formal and referential conventions. While I restrict my attention just to Cauchy's introduction, there are ample grounds for further analysis within the broader corpus of prefatory enunciations of his period and discipline.

beginning in 1805 at the age of 16, and his reception as a new faculty member from 1816.[6] In brief, Cauchy's studies were marked by a massive reorganization and militarization of the École by Napoléon's government which coincided with an increased emphasis on analysis as a cornerstone of the engineering curriculum. Taught by Sylvestre Lacroix with the help of André-Marie Ampère, his *répétiteur*, the analysis course was centered on a geometrically-derived limit concept.[7]

This basis for the course was challenged not long after Cauchy's departure in an 1810 official review, which criticized the curriculum as too theoretical and recommended the replacement of the limit concept with those of infinitely large and infinitely small quantities. These latter were deemed more suitable to the practical needs of engineers, and they were joined by more applied methods in the regular curriculum. The École's politically active student body was disbanded by royal ordinance in April, 1815, following Bonapartist demonstrations on the heels of the 1814 Bourbon Restoration, and Pierre Simon Laplace was charged with reorganizing the school for its 1816 reopening. Among Laplace's politically deft moves was replacing the politically liberal analysis instructor Poinsot with the ardent Catholic royalist Cauchy, who was joined by his mentor and ally Ampère as an instructor and by fellow Catholic Gaspard-Gustave de Coriolis as his *répétiteur*. Cauchy's early years as professor of analysis were marked by sweeping changes to the curriculum and sweeping criticisms from the École's authorities. Chief among his critics was the politically liberal professor of geometry and applied analysis François Arago, who singled out Cauchy's reversion to the limit concept in place of infinitely small quantities as particularly egregious, though the stubborn and irreverent Cauchy gave his colleagues many further grounds for complaint.

It is in this light that Cauchy begins his introduction with a deferential gesture, thanking his elder-statesman colleagues Laplace and Siméon-Denis Poisson for having been "willing to guide my first steps in the scientific profession, and . . . having testified to the desire to see me publish the Course in Analysis."[8] On the surface, his opening sentence is a simple and humble acknowledgement of his professional debt to these two prominent mathematicians. Laplace's influence at the École extended well beyond his role in the 1816 reorganization; he then served as one of the school's principal examiners and remained its vocal advocate as a recently elected member of the Académie Française.[9] Cauchy first encountered Laplace through his father's involvement with the senate formed

[6]For further context within Cauchy's career see Iacobacci (1965), Belhoste (1985), and Bottazzini (1992), and within the history of the École see Belhoste (1994, 2001), Chatzis (1994), as well as Gillispie (2004, pp. 520–540). Dahan Dalmedico (1992) exhaustively chronicles Cauchy's role in reshaping the French scholarly elite during this period, while Fox (1974), Grattan-Guinness (1990), and Fraser (1997) give related histories less focused on Cauchy. Finally, Alder (1995, 1999), Rusnock (1995), and Gillispie (2004, pp. 652–655) place Cauchy's mathematical context in wider revolutionary trends with respect to governance and education.

[7]On Lacroix's limit concept and the analysis curriculum of this period, see Gilain (1989, p. 5) and Richards (2006, pp. 710–711). In particular, Lacroix defines limits in terms of the vanishing magnitudes of successive differences, invoking a vocabulary shared between geometry and analysis.

[8]Quelques personnes, qui ont bien voulu guider mes premiers pas dans la carrière des sciences, et parmi lesquelles je citerai avec reconnaissance MM. *Laplace* et *Poisson*, ayant témoigné le desir de me voir publier le Cours d'analyse de l'École royale polytechnique, ... (Cauchy, 1821, p. *i*). All translations are my own. Since this paper's initial preparation, Bradley and Sandifer (2009) have published a full English translation of the Course's second edition.

[9]On Laplace's prominence in this period, see Gillispie (1997, pp. 167–168, 177, 277).

after the uprising of 18 Brumaire, of which Laplace was a member, and Laplace had indeed helped to shape Cauchy's earliest steps in the discipline. Poisson was an early standout at the École and was hired immediately upon his graduation in 1800. By the time of Cauchy's publication, Poisson had begun serving on the *Conseil royal de l'Université*, a post of national importance, and his own *Traité de mécanique* was a prominent attempt at pedagogical reform (Arnold, 1981, pp. 23–24; Costabel, 1981, pp. 3–4, 9). Poisson, too, had helped introduce Cauchy at a very early age to the Parisian mathematical circles. More importantly, however, both Laplace and Poisson were highly respected and influential figures within the mathematical profession. By citing them, Cauchy vests himself and his work in the authority of these much-better-esteemed professors. Given his controversial status within the École, Cauchy could not dispense with their support, real or symbolic.

The opening sentence also contains the first of two expressions in the first paragraph of the *usefulness* of his text. Here, the Course is being published "for the greater utility to students"[10] and further down Cauchy claims to "have presented developments which could be useful to Professors and to Students of the Royal Colleges."[11] At the time of the Course's publication, the opacity of Cauchy's lectures was a major point of contention between the instructor and the councils at the École responsible for evaluating both the curriculum and his performance presenting it. Although his text was distinctly devoid of discussions specific to engineering, it was still to be useful to students of the subject as a foundational text. Indeed, its hoped-for utility extended not just to professors and students, but "to those who would make a special study of analysis" (ibid.). The same text could serve two functions for two audiences. It could provide the analytic foundations necessary for advanced study in any field of engineering, and it could form the basis of a concerted study of analysis in its own right.

This vision for the text directly opposed the contentions of Arago and his allies, who insisted that the Course's emphasis on foundational concepts did nothing but distract from analysis's useful applications. The paragraph responds to Arago in one further way. Part of Arago's criticism of Cauchy's course is that it failed to adhere to explicit instructions from the École's administration to make infinitesimals, rather than limits, the conceptual core of the course.[12] Where Cauchy writes that he has "been unable to dispense with making the principal properties of infinitely small quantities known, properties which form the basis for infinitesimal calculus,"[13] the remark may well be colored by a shade of resentment. He has been unable to dispense with infinitesimals because of explicit mandates from the École, against his better judgement. Indeed, Gilain (1989, p. 12) has argued that the section introducing infinitesimals was most likely inserted late in the preparation of Cauchy's manuscript.

Thus, in a few turns of phrase at the start of his introduction, Cauchy seeks to answer a host of outstanding questions about his politically opportune appointment, his ability

[10]...pour la plus grande utilité des élèves. (Cauchy, 1821, p. *i*)

[11]...j'ai présenté des développemens qui peuvent être utiles soit aux Professeurs et aux Élèves des Colléges royaux, soit à ceux qui veulent faire une étude spéciale de l'analyse. (Cauchy, 1821, p. *ii*)

[12]Cauchy's use of infinitely small quantities has elicited much discussion in the secondary literature, particularly in analyses of Cauchy's errors and omissions. See, e.g., Laugwitz (1987).

[13]En parlant de la continuité des fonctions, je n'ai pu me dispenser de faire connaître les propriétés principales des quantités infiniment petites, propriétés qui servent de base au calcul infinitésimal. (Cauchy, 1821, p. *ii*)

to teach students of engineering, and his willingness to engage with members of the French mathematical establishment, even as he was moving quickly to supplant them.

3. Geometries and algebras

Cauchy's introduction is best remembered for its oft-quoted statement of method, which opens the second paragraph:

> Regarding methods, I have sought to give them all the rigor one requires in geometry, in such a way as never to resort to reasons drawn from the generality of algebra.[14]

The conventional interpretation of this passage is that Cauchy here contrasts the unrigorous use of algebraic formalisms by his mathematical predecessors, particularly in the eighteenth century, with the Euclidean geometric ideal of purely deductive mathematics.[15] This is of course true, but something is missing if one leaves the interpretation there.

The crucial distinction between geometry and algebra, for Cauchy, signified far more than the contrast of unguarded formalism against rigorous foundations. The difference comes from a double meaning, dating to the Early Moderns, of both terms.[16] In the second act of Cauchy's introduction, geometry refers not just to the deductive *more geometrico*, or geometric method, but also to the Euclidean study of quantities in space.[17] By the turn of the nineteenth century, the two meanings of geometry had largely parted and gone their separate ways. Philosophers ranging from Descartes to Hobbes to Spinoza applied the geometric method to nature, God, the mind, law, government, and ethics. Meanwhile, the analytic geometry of the seventeenth century had wedded the study of space and quantity to new algebraic methods that had less and less to do with what was seen as Euclid's antiquated deductive approach (Mahoney, 1980; Richards, 2006).

While there existed a wide variety of views concerning algebraic methods among Cauchy's contemporaries and predecessors, there emerges in Cauchy's pivotal statement of method a similar double meaning for algebra, one which was certainly consistent with a great many of the meanings then in circulation. Invoking "the generality of algebra," Cauchy drew on an image of algebra as an art of abstract formalisms. Like geometry, this algebra had both philosophical and mathematical registers. Mathematically, algebra meant using abstract formal symbols and manipulations to solve mathematical problems. From the Early Moderns onward, algebraic manipulations came to be characterized by their detachment from the physical interpretations at the base of Euclidean geometry, with algebraic formalism viewed as an "instrument of discovery in its own right" (Grosholz, 1980, p. 161; Mahoney, 1980, p. 142). Where the abstractions of Euclidean proofs owed their allegiance to in-principle realizable physical-geometric entities,

[14] Quant aux méthodes, j'ai cherché à leur donner toute la rigueur qu'on exige en géométrie, de manière à ne jamais recourir aux raisons tirées de la généralité de l'algèbre. (Cauchy, 1821, p. *ii*)

[15] Grabiner (1981, pp. 30, 164) gives an exemplary statement to this effect.

[16] See Mahoney's (1980) overview of the emergence of algebra in Europe during the seventeenth century and Høyrup's (1996) study of the Early Modern European appropriation of Euclidean geometry and its philosophical principles.

[17] I use the modern phrase 'quantities in space' to encompass the variety of geometric terms used by Cauchy and his contemporaries, including *grandeur* and *quantité*.

algebraic formalisms were unmoored from whatever meaningful problem brought them into existence. Without constant regard for their geometric meaning, such formalisms easily could and routinely did produce such physical absurdities as negative or imaginary quantities.[18]

Algebra's philosophical meaning was closely tied to its mathematical one. No one spoke of a *more algebraico*, but algebra's mathematical contrast to geometry placed it in opposition to geometry's philosophical valuation as well. Algebraic methods could be seen as powerful and fruitful, but lacking in the foundational geometric rigor which, at least in the French eighteenth century, was widely regarded as superfluous and pedantic (Mahoney, 1980, p. 147; Richards, 2006). To an algebraic mathematician, Richards argues, the "rigor of geometry" simply got in the way. The same formalisms that allowed for negative numbers and other monstrosities proved their ultimate worth in the mathematical truths they brought to light. Algebra thus became a pragmatic method—a means to ends that would speak for themselves. As a philosophy, algebra meant doing mathematics for the sake of what it could accomplish and without care for the petty details of its premises. Below, we shall see evidence of this view in the writings of Cauchy and his predecessors.

It is clear that Cauchy's introduction sought to replace a philosophical algebraism with a philosophical geometrism. Less clear is the extent to which this substitution relied on a similar substitution in the mathematical realm. This reliance is hidden in large part because Cauchy's is, on the surface, a thoroughly algebraic text in its mathematics. It is, for instance, utterly devoid of diagrams, which are a hallmark of geometric exposition. But mathematical geometrism enters nonetheless. To see how it does so in the introduction, it will help to reconsider a canonical example from the mathematical exposition.[19]

3.1. Geometry in the Intermediate Value Theorem

Grabiner's classic study of the origins of Cauchy's program of rigor takes his proof of the Intermediate Value Theorem as a hallmark of his supposedly algebra-rooted approach (Grabiner, 1981, pp. 49, 71).[20] Roughly stated, the Intermediate Value Theorem says that if a continuous function moves from one value to another, it must at some point assume each value in between. Grabiner translates a proof from Cauchy's third appendix. This closely resembles proofs of the Intermediate Value Theorem that found favor in the second half of the nineteenth century and continue to be taught today. There, Cauchy describes the subdivision of an interval into smaller and smaller parts until a point can be found where the continuous function exactly achieves the value of interest. But this is not the only proof of the theorem in Cauchy's text.

[18]This is not to say that there were not physical interpretations at that time for the products of these formalisms. Rather, they lacked the *a priori* sensibility typically associated with the positive magnitudes of Euclidean geometry: one can give a physical interpretation to -1 (or its square root) but this does not help one to draw a triangle with that area.

[19]The argument that follows should not be read as an attempt to adjudicate the ultimate logical or mathematical status of Cauchy's Course. Below, I offer a geometric and intuitive reading of Cauchy's introduction in light of a strong tradition in the history of mathematics of understanding the contents of his Course to be anything but geometrical and intuition-based. In so arguing, I wish to stress the often surprising affronts to conventional interpretations that a close examination of important texts like Cauchy's can provide.

[20]Cf. Boyer (1968, p. 563) for Cauchy's purported algebraism.

In Grabiner's (1981, p. 71) words, the proof in the appendix shows that "Unlike his predecessors, Cauchy was not trying to approximate a root, but to prove its existence." Grabiner does acknowledge that the proof was based on a well-established approximation procedure.[21] Cauchy himself seems to side with this latter reading, describing the proof as "furnishing the numerical solution of the equation $f(x) = b$"[22] and placing it in an appendix on numerical methods whose main focus is the numerical determination of algebraic roots. If the primary purpose of his more famous proof of the Intermediate Value Theorem was, contrary to its later use, merely to aid in numerical approximation (a task Cauchy deemed secondary to the work of rigorous mathematics), then it behooves one to look more closely at the theorem's proof in the main body of Cauchy's text.

Cauchy's first proof of the Intermediate Value Theorem is the fourth proof he presents after giving his definition for continuity. It follows what Grattan-Guinness (1970, p. 377) dismisses as a "naive geometrical argument." In plain prose, Cauchy describes "the curve which has for its equation $y = f(x)$" in the plane and the level line $y = b$ (Cauchy's term is *droite*, the same term used for straight lines in Euclidean prose) corresponding to the value b in question.[23] To complete the proof, Cauchy observes that the level line falls by hypothesis between the vertical coordinates of the function at either end of the interval under consideration, so "the above mentioned curve cannot but meet [it] in the interval."[24] The whole discussion takes less than a page.

The first proof is remarkable precisely because it utterly fails to comply with the standards of rigor later associated with Cauchy. It does not invoke his definition of continuity. It uses analytic language only incidentally. And yet nowhere does Cauchy imply that this first proof is in any way inferior to the "purely analytic" one in the appendix. Indeed, the textual priority of the proof in chapter two would seem to suggest that one should rather turn to Cauchy's "naive geometrical argument" for insight into the particular form of rigor he aimed to espouse.

Grattan-Guinness (1980, p. 111) points out that this proof is in keeping with the contemporary custom of describing functions as though they were monotonically increasing.[25] More importantly, Cauchy's main proof describes the function in question as though it subsists *in the Cartesian plane* and, as Benis-Sinaceur (1973, p. 107) observes, comports to thoroughly geometric intuitions—a practice that can be found elsewhere in Cauchy's pedagogical writings as well.[26] Fundamentally, the theorem holds for Cauchy because, when translated into geometric terms, it describes two curves which must necessarily cross—the continuous function and the level line $y = b$. No further argument is needed. The formal definition of continuity need not even be invoked. What was impor-

[21] The point is also made by Freudenthal (1971, pp. 382–383) and Katz (2009, p. 773).

[22] . . . fournir la résolution numérique de l'équation $f(x) = b$. (Cauchy, 1821, p. 44)

[23] Pour établir la proposition précédente, il suffit de faire voir que la courbe qui a pour équation $y = f(x)$ rencontrera une ou plusieurs fois la droite qui a pour équation $y = b$ . . . . (Cauchy, 1821, p. 44)

[24] . . . ce qu'elle ne peut faire sans rencontrer dans l'intervalle la courbe ci'dessus mentionnée. (Cauchy, 1821, p. 44)

[25] Cauchy nowhere stipulates that the curve must be monotone, nor does it need to be for his argument to follow (to the extent that it follows for the monotone case). Nevertheless, in light of the context in which Cauchy wrote and the historiographical and mathematical contexts in which he would later be placed, it remains significant that Cauchy makes no effort to describe the curve as anything other than monotonic. Cf. Lakatos (1979, p. 129), who characterizes a variety of interpretations of functions and their continuity from Cauchy and his contemporaries.

[26] For instance, Cauchy (1823, p. 8).

tant for Cauchy was "the rigor of geometry" in the face of the "generality of algebra": his proof by geometric intuition could even stray from the geometric method (and, in particular, the citation of antecedent propositions or definitions) because its evidence was so firmly rooted in the mathematical sense of geometry.

In the light of his main proof, Cauchy's proof in the appendix appears discernibly different as well. One starts to notice the subtle encroachments of geometric thinking in the "purely analytic" exposition, such as his free use of the concept of "opposite sign"—something Katz (2009, p. 733), for instance, conveniently elides by adding the criteria '$f(\alpha) < 0$ and $f(\beta) > 0$' in his recent presentation of Cauchy's argument. The proof proceeds by constructing two sequences of x-values narrowing in on a single limit but having opposite signs, a completely unproblematic procedure when one imagines a monotonic section of a function crossing the x-axis but one whose certainty of success is much less obvious when the function's sign oscillates rapidly near the limit of the x-sequences. The limit concept Cauchy invokes to close the proof is one steeped in geometric intuition (Benis-Sinaceur, 1973, p. 109).

3.2. Sin and expiation

Cauchy's new rigorous mathematics, then, was to be vested in both senses of geometry. Having seen the manifestation of this program in the Intermediate Value Theorem, one can more easily discern its appearance in what follows in Cauchy's introduction: a list of faults found in the mathematics of his predecessors and a list of restrictions "which perhaps appear a little harsh at first"[27] but which serve to "make all uncertainty disappear."[28] His two-tiered presentation takes the form of a Catholic confession, moving from an accounting of sins to a program for their absolution.

Belhoste (1991, pp. 136–142) documents Cauchy's fierce reactionary Catholicism and the inquisitorial reputation it earned him among the French academic elite. Indeed, Cauchy exhibited few reservations about drawing religious and political controversies into his assessments of others' scientific work. Regarding his own work, Cauchy's insistence on doctrinal authority helps to explain the manner in which he aimed to restore rigor to analysis. Specifically, Cauchy's list of sins and remedies echoes the confessional doctrine of the Council of Trent, which stressed the detailed and specific enunciation of sins so that through a knowledge of their causes the priest may apply an appropriate penance (Bernard, 1911, p. 904; Ortolan, 1911, pp. 953–954). If the shortcomings of his predecessors' mathematics could be enumerated and corrected, Cauchy could aspire to produce a right mathematics governed by rigorous discipline, and thus to mirror the righteous Catholic governed by religious discipline.

This is not to argue that Cauchy's Catholicism was the only (or even the primary) influence on his presentation in this section of the introduction. The form of Cauchy's enunciation coexists with a wide variety of religious tropes and styles, and was by no means exclusive to the particular context of the Catholic confession. But taken in the context of Cauchy's well-known religious commitments, this simple structural observation affords an enriched view of the epistemic maneuvers to come in the introduction's third

[27] . . . je me suis vu forcé d'admettre plusieurs propositions qui paraîtront peut-être un peu dures au premier abord. (Cauchy, 1821, p. *iv*)

[28] . . . je fais disparaître toute incertitude. (Cauchy, 1821, p. *iii*)

act. In the present passage, Cauchy urges mathematicians to restrict their claims and their actions in order to validate their results. In the third act, not just mathematicians but all would-be producers of knowledge must likewise conform to modes of knowing that constrain what each can say to the other. This latter assertion, as Belhoste notes, has strong roots in the Restoration-era ultramontane orthodoxy that shaped Cauchy's strongly held views on knowledge and authority.

Having diagnosed the overarching problem of the dangerous generality of algebra in his opening sentence, Cauchy specifies the particular sins manifest in that generality. They involve "the passage from convergent to divergent series, and from real quantities to imaginary expressions" and "attribut[ing] to algebraic formulae an indefinite extension." "Most of these formulae," explains Cauchy, "subsist only under certain conditions, and for certain values of the quantities they contain." He continues,

> By determining these conditions and these values, and in fixing in a precise way the meaning of the notations of which I avail myself, I make all uncertainty disappear; and so the different formulae no longer present anything more than relations between real quantities, relations which are always easy to verify by the substitution of numbers for the quantities themselves.[29]

Equipped with a geometric reading of the Intermediate Value Theorem, Cauchy's list of sins is subtly transformed. In the conventional reading, each sin is a straying by formal algebra from the foundations of the *more geometrico*. In each case, an innocuous formalism for a convergent series or real variable loses sight of its conditions of validity and comes to represent, for the incautious algebraist, a divergent series or imaginary expression.[30] Formulae, for Cauchy, need always to remain loyal to their foundations and are justified only in specific domains of validity.

His is a problem about foundations and the *more geometrico*, yes, but it is also a problem about mathematical geometry. It is not enough to fix domains of validity for one's formalisms; one must do so to preserve their relevance to "relations between real quantities," the very relations underlying the Euclidean study of magnitude.[31] The generality of algebra is insidious only insofar as it lets one stray from the study of geometrically valid quantities. Imaginary quantities, with their squares of negative area, and divergent series, with their untamed (even infinite) sums, could not be accredited in a properly geometric mathematics. Fixing algebraic conditions for the validity of algebraic formalisms, Cauchy guarantees their geometric certainty without having to resort to the tedium of actual Euclidean geometric proofs.

The conditions themselves bear out this geometric interpretation. Each statement in this section of the paragraph is a specific act of policing the grounds of a certain type of

[29]... dans le passage des séries convergentes aux séries divergentes, et des quantités réelles aux expressions imaginaires .... [Elles] tendent à faire attribuer aux formules algébriques une étendue indéfinie, tandis que, dans la réalité, la plupart de ces formules subsistent uniquement sous certaines conditions, et pour certaines valeurs des quantités qu'elles renferment. En déterminant ces conditions et ces valeurs, et en fixant d'une manière précise le sens des notations dont je me sers, je fais disparaître toute incertitude; et alors les différentes formules ne présentent plus que des relations entre les quantités réelles, relations qu'il est toujours facile de vérifier par la substitution des nombres aux quantités elles-mêmes. (Cauchy, 1821, pp. *iii-iv*)

[30]A related interpretation is given by Benis-Sinaceur (1973, p. 105).

[31]Section 3.3, below, discusses just how close Cauchy's 'real quantities' were to their Euclidean counterparts.

perilous formalism where it might stray from geometric validity.[32] Divergent series cannot be said to have a sum, for they no longer represent the stable accretion of geometric magnitudes to which the sum ought to correspond. Imaginary equations can only be said to have meaning as pairs of coupled equations between real quantities, and so become just another form of Cauchy's (and Euclid's) prized relations. If an equation violates the conditions of its formulation, it must either be discarded or furnished with a new set of conditions.[33] In short, algebra must be *both* mathematically and methodologically geometric.

Cauchy finishes the second paragraph of his introduction by stressing that "the happy necessity of putting more precision in theories" can also "furnish several subjects of research which are not without importance" in themselves.[34] Not only do his restrictions on the generality of algebra rescue its geometric validity, but they are also genuinely productive. A good Catholic, Cauchy insists that proper restraint gives one the certitude attendant to knowing one's methodological place as well as the further reward of providing new forms and avenues of production, inquiry, and enlightenment.

3.3. The two faces of algebra

The heart of Cauchy's critique of the generality of algebra is baldly stated just after the famous statement of method:

> Reasons of this type ... cannot be considered, it seems to me, but as inductions so as to sometimes apprehend the truth, but which agree little with the vaunted exactitude of the mathematical sciences.[35]

Algebra's most heinous sin, it seems, is that it is useful, that it is capable of producing the right answers even though they are justified with the wrong reasons. Whether the target of the barb in Cauchy's statement of method was Swiss mathematician Leonhard Euler (Gilain, 1989, p. 24), Cauchy's French forebear Joseph-Louis Lagrange (Bottazzini, 2001, p. 33), or Laplace and Poisson (Belhoste, 1991, p. 51), Cauchy's criticism is clear: these great mathematicians may have obtained genuine results, but they were not performing genuine mathematics.

Here, Cauchy invokes the dual status of algebra at its most basic. Algebra is a powerful extension of geometric methods. Cauchy explains that

> The use of symbolic expressions or equations is often a means of simplifying the calculations and of writing results which are quite complicated in appearance in an abridged form.[36]

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[32] Benis-Sinaceur (1973, pp. 102–103) offers a similar reading.

[33] This precept is not itself explicitly geometric. But in the context of the others it has a clear geometric interpretation: if an equation loses reference to its initial geometric referents, something must be done to restore its access to a non-formal geometric foundation.

[34] ... les propositions de cette nature [the conditions just discussed], entraînant l'heureuse nécessité de mettre plus de précision dans les théories,... fournissent plusieurs sujets de recherches qui ne sont pas sans importance. (Cauchy, 1821, p. *v*)

[35] Les raisons de cette espèce, ... ne peuvent être considérées, ce me semble, que comme des inductions propres à faire presentir quelquefois la vérité, mais qui s'accordent peu avec l'exactitude si vantée des sciences mathématiques. (Cauchy, 1821, pp. *ii–iii*)

[36] L'emploi des expressions ou équations symboliques est souvent un moyen de simplifier les calculs, et d'écrire sous une forme abrégée des résultats assez compliqués en apparence. (Cauchy, 1821, p. 173)

But algebra is not just powerful. It is also an impostor. Algebraic techniques allow one to derive results with genuine geometric meaning. But by moving the mathematician one step away from geometric objects themselves, algebra allows him to stray from geometry and produce statements with no geometric meaning whatsoever. Algebra works "sometimes," but without the utmost of vigilance it is easy to lose track of which times those are.

The situation is one of grammatological supplementarity.[37] In Cauchy's view, algebra begins as a helpful aid to geometry. As its power and influence grows, however, algebra comes to displace geometry, usurping its mathematical priority by exposing its relative impotence. Cauchy echoes a common historical refrain. His reassertion of the primacy of geometry is a desperate bid to win control over an algebraic golem run amok.

The sentiment is well encapsulated in Cauchy's first Note at the end of the course. Algebra, he writes,

> is an artifice that augments the resources of analysis; but whenever one wants to use it, it is necessary to take heed of the following conventions.[38]

Seeking "to avoid every type of confusion in algebraic language and writing"[39] becomes the object of a number of correctives meant to keep the artifice of algebra true to the geometrically meaningful results it would elucidate, including the ones listed in the introduction, discussed in the previous section. In his Preliminaries and in Note I, Cauchy reduces algebraic formalisms to their fundamental geometric referents, defining numbers and quantities in terms of operations on positive magnitudes of a fixed type (that is, dimension)—the same magnitudes at the core of Euclidean geometry (Cauchy, 1821, pp. 1–3 et seq., 403–404 et seq.). That Cauchy spends some seven pages of the Course on the distinction between positive and negative numbers indicates just how important it was for him to rid algebra of the most basic sins of its formalism.

But algebra's straying from Euclidean geometry is a double straying. It is only possible when the algebraist also loses sight of the *more geometrico* itself. Thus, Cauchy both insists on the primacy of relations between real quantities *and* puts his methodological correctives in the form of foundational rules in the axiomatic tradition. The only way to ensure that algebraic methods produce geometric results (in the mathematical sense) is to develop the algebraic exposition according to the *more geometrico*, not even taking such basic notions as an algebraic variable or constant for granted.[40] Without such a progression from fundamentals, analysis could never have access to the certainty built into geometric rigor. He has been "forced to admit several propositions"[41] because the algebra's referential loyalty to the objects of mathematical geometry could only come from a simultaneous insistence for algebra on a new fealty to philosophical geometry.

Importantly, however, Cauchy did not address the algebraic heresy by doing away with the method altogether. Engineers at the École would still need algebraic methods,

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[37] For a foundational statement, see Derrida (1974).

[38] Au reste, rien n'empêche de représenter les quantités par de simples lettres aussi bien que les nombres. C'est un artifice qui augmente les ressources de l'analyse; mais lorsqu'on veut en faire usage, il est nécessaire d'avoir égard aux conventions suivantes. (Cauchy, 1821, p. 404)

[39] Pour éviter toute espèce de confusion dans le langage et l'écriture algébriques, ... (Cauchy, 1821, p. 1)

[40] E.g. Cauchy (1821, p. 4).

[41] Cauchy (1821, p. *iv*), op. cit.

and Cauchy himself could not have doubted the utility of algebra's formal approach. He remedied the sins of algebra with rules of practice that were fundamentally algebraic, offering algebra a geometric redemption in algebraic terms. In so doing, he subtly rewrote the rules of the *more geometrico*. His regrounding of algebra admitted the possibility that formal algebraic rules (in Cauchy's case, those derived from mathematical geometry) could serve as the basis for a rigorous mathematical practice which was practically and methodologically rooted not in spatial geometry but in formal algebra. This possibility was taken up by the foundationalists of the second half of the 19th century, in what Lancelot Hogben (1960) later called "The Great Biopsy." Where Cauchy "sought to give them all the rigor one requires in geometry, in such a way as never to resort to reasons drawn from the generality of algebra," those such as Richard Dedekind could "resolve to ... find a purely arithmetic ... foundation" and criticize their predecessors for "appeal[ing] to geometric notions or those suggested by geometry" (Dedekind, 1948, pp. 1–2).

Cauchy's geometric response, in both senses of geometry, to the generality of algebra served at least two broader social functions. First, it offered a means by which Cauchy could claim a well-founded certainty for his mathematics amidst the epistemic chaos Richards (2006) describes in post-Revolution France. Second, as Belhoste (1991, p. 213) suggests, moves such as Cauchy's helped mathematics lead a movement across many disciplines in the first half of the 19th century increasingly to assert control over their respective content and standards.[42] Mathematical rigor has an inward-looking role of shoring up the process of knowledge creation, but this shoring-up cannot be divorced from its necessary effect of limiting and regulating access both to the tangible institutional grounds of the discipline and to the epistemic grounds on which knowledge claims can be made. It would be misleading to say that Cauchy's mathematics was influential *despite* its being difficult, for its difficulty was a necessary social condition for the epistemic work that would win him fame.[43] Right method is more than merely a restriction of cognitive possibilities. It is a restriction of participatory possibilities as well.

4. Staging mathematics

Where Cauchy's second act began to draw walls around a specific domain of mathematical practice, his third act made these divisions sharp and clear. In the second act, Cauchy sought to narrow the range of what could produce mathematics. To this, in the third act, Cauchy adds prohibitions on what mathematics could produce.

There was an integral connection between the inputs and outputs of mathematical practice. Cauchy was responding not just to an irresponsible recourse to the generality of algebra. He was also faced, in the mathematics of his rivals, with a worldview where mathematics could and should speak to anything and everything. Powerful but ungrounded methods were justified when they could draw their legitimacy from the worldly results they produced. Only by taking away this sphere of application from

[42] See also Alexander (2006) and Porter (1995, part 3), which discuss two related narrative dimensions of this disciplinary transformation.

[43] Belhoste (2003, p. 231) elaborates the social function of the difficulty of Cauchy's mathematics in the context of the École. In particular, he points to the elitist function of Cauchy's method, whose effect was to erect ever-greater barriers to mathematical participation by non-Polytechnicians.

genuine mathematics could Cauchy replace it as a regime of justification with his own foundational program.

To begin to understand the origins and purpose of Cauchy's walling-in of mathematics, consider Belhoste's contention that Cauchy's Catholic epistemology gave the mathematician a bipartite model of truth, with moral truths of the Church contrasted against the scientific truths of the natural sciences. Belhoste (1991, pp. 216–217) claims that Cauchy's introduction places his mathematics squarely at the opposite end of the Catholic moral order as an exemplar of scientific truth. In fact, Cauchy goes much farther than this simple two-part distinction, though it no doubt grounded his decision to designate different ways of knowing for different spheres of knowledge.

Cauchy writes explicitly of the ties between the production and application of mathematics to start his third paragraph:

> For the rest, if I have sought, on the one hand, to perfect mathematical analysis, on the other hand, I am far from pretending that this analysis should have to suffice for all the rational sciences.[44]

He explains that to be successful in the natural sciences one can only proceed by "observing events and then submitting the evidence to calculation."[45] One must take care in reading the phrase which follows. Cauchy begins his next sentence with a "But," suggesting that Belhoste may be right in grouping mathematics with the natural sciences. "But," Cauchy writes, "it would be a grave error to think that one only finds certainty in geometric demonstrations, or in the testimony of the senses."[46] Cauchy preserves a role for mathematics in the natural sciences, but he simultaneously construes them as fundamentally different. He contrasts "geometric demonstrations" with "the testimony of the senses," implying that the latter have no place at all in mathematics, while the former may still be of use to other sciences insofar as they support calculations.

The world can have nothing to say about mathematics, but at least in this limited capacity mathematics can still tell something of the world. Cauchy goes on to make clear just how limited a capacity this is. In so doing, he establishes a fundamental difference between the truths of mathematics and the truths of society. Here is his pivotal sentence in full:

> But it would be a grave error to think that one only finds certainty in geometric demonstrations, or in the testimony of the senses; and although no person until now has tried to prove by analysis the existence of Augustus or of Louis XIV, every sensible man will agree that this existence is as certain for him as the square of the hypotenuse or Maclaurin's theorem.[47]

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[44]Au reste, si j'ai cherché, d'une part, à perfectionner l'analyse mathématique, de l'autre, je suis loin de prétendre que cette analyse doive suffire à toutes les sciences de raisonnement. (Cauchy, 1821, p. *v*)

[45]. . . à observer les faits et à soumettre ensuite les observations au calcul. (Cauchy, 1821, p. *vi*)

[46]See full quotation below.

[47]Mais ce serait une erreur grave de penser qu'on ne trouve la certitude que dans les démonstrations géométriques, ou dans le témoignage des sens; et quoique personne jusqu'à ce jour n'ait essayé de prouver par l'analyse l'existence d'Auguste ou celle de Louis XIV, tout homme sensé conviendra que cette existence est aussi certaine pour lui que le carré de l'hypotenuse ou le théorème de *Maclaurin*. (Cauchy, 1821, p. *vi*)

It would have been epistemic suicide for Cauchy to claim that the only truths were those of geometry. Instead, he says something much more powerful, perfectly of a piece with the disciplinary enclosure movement he helped spur. For Cauchy, the truths of geometry compose just one class of a wide variety of truths: mathematical, natural, moral, political, and so forth. Before exploring how Cauchy develops this notion, some of his rhetorical choices deserve special remark.

It is no coincidence that Cauchy's example of a social truth invokes iconic figures whose respective command of the French monarchy and Roman Empire could not be doubted. Indeed, there could be no better example for Cauchy of the possible transcendence of social truths—every schoolchild in France would have known of these two giants. It was not just that nobody could doubt the existence of these two rulers. Rather, the political facts of their existence were *just as certain* as the truth of their mathematical counterparts.

With the square of the hypotenuse, Cauchy refers to the Pythagorean theorem, an archetypal example of ancient Euclidean geometry corresponding to his choice of an archetypal Classical monarch. To this, he pairs a mathematical-analytic theorem of Maclaurin's from the century just prior to his own. The choice of the Scottish mathematician was not arbitrary. Cauchy was directly attacking his immediate forebears, particularly in France, and it might have seemed contradictory to turn around and hold their work as an example of certain knowledge. Maclaurin was sufficiently well-known and respected among French mathematicians to make his theorem recognizable, but also sufficiently distant in both space and time to dissociate his mathematics from that which Cauchy criticized.

Moreover, Maclaurin's results regarding the representation of functions by power series were a matter of important dispute among Cauchy and his contemporaries. Lagrange had made such series the foundation of his theory of analytic functions, marking a sharp turn from what Richards characterizes as the narrative approach to mathematics in 18th-century France (Richards, 2006, pp. 709–710). His influential series-based method had found many adherents in the decades between Lagrange's *Théorie des fonctions analytiques* and Cauchy's Course, but it does not figure at all in Cauchy's foundational text. In the lectures Cauchy published two years later, Cauchy explicitly denounced power series as a foundation for calculus,[48] calling attention to the uncertainties accompanying nonconvergent series and devoting large portions of his 37th and 38th lectures to the convergence conditions for Taylor and Maclaurin series (Cauchy, 1823, pp. *v–vi*, 145–152).[49] His reference to Maclaurin allowed Cauchy to assert scholarly authority over a topical

[48]E.g.: ...je l'espère, que les principes du calcul différentiel, et ses applications les plus importantes, peuvent être facilement exposés, sans l'intervention des séries (Cauchy, 1823, p. *vi*). (...I hope that the principles of differential calculus, and its most important applications, can be easily stated without the intervention of series.)

[49]E.g.: Mais, malgré tout le respect que commande une si grande autorité [auteur de la *Mécanique analytique*], la plupart des géomètres s'accordent maintenant à reconnaître l'incertitude des résultats auxquels on peut être conduit par l'emploi de séries divergentes, et nous ajouterons que, dans plusieurs cas, le théorème de TAYLOR semble fournir le développement d'une fonction en série convergente, quoique la somme de la série diffère essentiellement de la fonction proposée (Cauchy, 1823, p. *vi*). (But, despite all the respect such a great authority [Lagrange] commands, the majority of mathematicians now agree to admit the uncertainty of results which one might achieve by the use of divergent series, and we will add that, in several cases, Taylor's theorem seems to produce the development of a function as a convergent series, even though the sum of the series essentially differs from the proposed function.)

result. Indeed, Maclaurin series were among the chief formalisms to be tamed by the proscriptions discussed above.

Cauchy's citation of Maclaurin also had specific methodological overtones, invoking positions championed by Maclaurin which were driven by a religious and ideological fervor to match Cauchy's own (Grabiner, 2004). For while Maclaurin's discoveries were much lauded in turn-of-the-century France (Grabiner, 1997), the same cannot be said of his methods. Consider his treatment, indicative of his reputation by the end of the 18th century, in an 1802 edition of Montucla's[50] widely-read *Histoire des mathématiques*:

> His prolixity was somewhat superfluous. We can do nothing, in France, but laugh at the attack of that ingenious visionary .... M. Maclaurin has given ... proofs multiplied by his taste for ancient geometry.[51]

Or again, explaining that Newton and his contemporaries had been charged with using false or incomplete notions, he adds

> It is true that at the time of this last attack, Maclaurin defended them and established their principles on proofs in the manner of the ancients. But what tiring prolixity, what circle of of reasoning is he not obliged to employ![52]

Cauchy found in Maclaurin a rigorous crusader before his time, one ardently and famously loyal to the methods of the ancient geometers.

Just as Louis XIV was no ordinary king of France, Maclaurin's was no ordinary theorem. Cauchy picked the cream of each crop to represent their respective domains of knowledge. And yet, for the particular claims Cauchy made—about the existence of kings and the truth of theorems—it would seem that any sufficiently well-known monarchs or theorems would do. It remains to account for the citation of Louis XIV over, for instance, the ill-fated Louis XVI whose reign was overthrown by the French Revolution, or of Maclaurin's theorem over one from any of the leading lights of 18th-century French mathematics. To see why, it is necessary to turn to the rhetorical force of Cauchy's argument.

Cauchy's introduction, as Bottazzini reminds us, does not merely rest on its empirical claims. It is clear that in the previous section, where Cauchy makes an *empirical* list of the remedies he has enacted for his mathematics, the list serves also as a *normative* injunction for the mathematics of his colleagues, pupils, and other readers. It is not just that Cauchy has modified his mathematics in the ensuing text, but that all mathematicians should do the same. As a manifesto, the introduction's blurring of the empirical and normative allowed it to promulgate a dense catalogue of denunciations and reforms as though their truth were as natural a question as that of who governed France. Seen

[50]On this volume's authorship and its relation to the epistemological battles which Cauchy would later join, see Richards (2006, pp. 702, 707–708).

[51]Sa prolixité étoit assez superflue. Nous n'aurions fait en France que rire de l'attaque de cet ingénieux visionnaire, qui traitoit la geométrie des fluxions d'une coupable hérésie, et les Géomètres, d'incrédules. M. Maclaurin a donné d'ailleurs dans ce même ouvrage, des preuves multipliées de son goût pour la géométrie ancienne. (Montucla, 1802, p. 11)

[52]Il est vrai qu'à l'époque de cette dernière attaque, Maclaurin les défendit et établit leurs principes sur des démonstrations à la manière des anciens. Mais quelle prolixité fatiguante, quel circuit de raisonnemens n'est-il pas obligé d'employer! (Montucla, 1802, p. 270)

normatively, Cauchy's text argues not just that Louis XIV *was* the king of France but that he *should have been*, not just that Maclaurin's theorem *is* true but that it *should be* so: Louis XIV had an inherent (indeed, divine) right to govern; Maclaurin's theorem had an inherent (indeed, Euclidean) right to truthhood.

It is always possible to read Cauchy's claims empirically—indeed, many of his foes would likely have done so—but such a reading would ignore the introduction's crucial rhetorical dimension. In the Manichean cosmology of the manifesto, citations of fact are never merely that. Nor, for that matter, are rhetorical pairings. Louis XIV and Augustus were not just two kings from the Classical and recent pasts, but two kings whose power and authority were grossly evident. Maclaurin's and Pythagoras's theorems were not just two mathematical results, but two mathematical results of iconic importance established through the Euclidean geometric tradition. The pairings show a corollary normative interpretation for the citations: kings should be powerful like Louis XIV, theorems should be promulgated and justified in the manner of Maclaurin's.

Louis XIV and Maclaurin's theorem continue to impart lessons in normative epistemology in the next sentence. "I will say more," Cauchy adds,

> the demonstration of this last theorem is comprehensible to a small number of minds, and the scholars themselves are not all in agreement on the extension one should attribute to it; whereas everyone knows very well by whom France was governed during the seventeenth century.[53]

There are, for Cauchy, different kinds of truths which call for different methods of verification employed by different people. Indeed, the 17th century king of France not only is at least as certain as Maclaurin's theorem, but this certainty is and *should be* accessible to a much broader public than that of mathematical truth. Reading Cauchy's statement in its manifesto context, Cauchy stipulates not just the empirical claim that far more people know French political history than can debate the extension of Maclaurin's theorem, but that this particular order of things should stand normatively as a model for the rest. For Cauchy, mathematical truth is not special because it is general. On the contrary, it is special because it is geometric, and geometry remains a science of the privileged.

In ceding ground to natural scientists, political philosophers, and historians, among others, Cauchy simultaneously reinforces the borders of mathematics. While the ruler of France may be a popular question, the validity of Maclaurin's theorem must not be. Cauchy's mathematics is not for everyone, nor does it aspire to be. A committed royalist, his comparison of Louis XIV and Maclaurin's theorem advances the principle that the most universal of truths cannot be entrusted to the most universal of arbiters. His mathematics was to be one of narrow application and narrow participation. Dispute was to be confined to elite scholarly circles, and questions of the moral order of society were to be rigorously excluded. Cauchy elaborates:

> That which I say here of historical fact can equally apply to a multitude of questions, religious, moral, and political. We must thus be persuaded that

[53]Je dirai plus; la démonstration de ce dernier théorème est à la portée d'un petit nombre d'esprits, et les savans eux-mêmes ne sont pas tous d'accord sur l'étendue qu'on doit lui attribuer; tandis que tout le monde sait fort bien par qui la France a été gouvernée dans le dix-septième siècle .... (Cauchy, 1821, p. *vi*)

there exist other truths than the truths of algebra, other realities than sensible objects.[54]

What emerges is not a bipartition of knowledge between God and man but a pluripartition of knowledges amongst a rigidly divided field of scholars and practitioners. Cauchy's academy was to be built on the principle of different truths for different disciplines.

It was to be crucial that these truths and methods did not become entangled:

> Let us cultivate the mathematical sciences with ardor, without wanting to extend them beyond their domain; and we will not imagine that one can attack history with formulae, nor give theorems in algebra or integral calculus in order to sanction morality.[55]

This last barb has been thought to be directed at Laplace and his use of the theory of probability to evaluate such things as historical testimony (Belhoste, 1991, pp. 51n27, 219), at Poisson (Grattan-Guinness, 1990, p. 714), or at theorists of a rational civil order such as Condorcet (Gilain, 1989, p. 11n59). Whatever its target, its motive is clear. All of these men, in the framework of Cauchy's introduction, engaged in a dangerous entanglement. Cauchy's predecessors had improperly conjoined mathematics and nonmathematics just as they had conflated geometry and algebra. The result was a groundless jumble whose political manifestation was the wave after wave of upheaval of the French Revolution and its aftermath. Revolutionaries had contaminated governance with democratic ideals; atheists misapplied their mundane reasoning to matters divine; mathematicians used their art beyond the narrow confines within which Cauchy insisted it be kept, both by overextending the outcomes of algebra and by pretending that history and politics could be attacked with formulae. Wrong mathematics, wrong religion, and wrong government came from the same methodological and epistemological confusion. Stepping out of the proper domain of geometry is, it is implied, one and the same with stepping out of the proper domain of subservience to the Crown. Each represents a departure from what is fundamentally right and true by chasing after imaginary ideals with powerful and dangerous methods.

5. Conclusion

The introduction concludes with further expressions of thanks toward Poisson, Ampère, and Coriolis, the latter two of whom are singled out for their mathematical contributions to the work. In one sense, this short epilogue represents a final layer of displacement to cap an intricate sequence of placements. Having positioned his course, geometry and algebra, and mathematics as a whole, Cauchy offers a personal note to displace his own authorship by acknowledging his debts to his colleagues.

There is a logic to Cauchy's overall structure. Step by step, the introduction telescopes in scale, starting with the locus of practice in the École and expanding through

[54]Ce que je dis ici d'un fait historique peut s'appliquer également à une foule de questions, en religion, en morale, en politique. Soyons donc persuadés qu'il existe des vérités autres que les vérités de l'algèbre, des réalités autres que les objets sensibles. (Cauchy, 1821, p. *vii*)

[55]Cultivons avec ardeur les sciences mathématiques, sans vouloir les étendre au-delà de leur domaine; et n'allons pas nous imaginer qu'on puisse attaquer l'histoire avec des formules, ni donner pour sanction à la morale des théorèmes d'algèbre ou de calcul intégral. (Cauchy, 1821, p. *vii*)

general methodological principles to their location in the field of knowledge as a whole. His introduction's conclusion reasserts the local work he has just begun, but by displacing rather than placing he makes the textbook part of the greater negotiation of knowledge and practice toward which the rest of his introduction builds. The telescoping structure of the introduction asserts that the Course fits within a certain orientation toward methodological rigor which is in turn situated in a broader epistemology, but it also conversely suggests the Course's place at the center of this epistemic universe.

Each part of Cauchy's positioning was necessary to establish his program of rigor. Consider, for instance, his subtle insistence on the mathematical sense of geometry in the second act. His commitment in the third act to rigorous mathematics as epistemically apart drew from and necessitated a mathematical foundation from the second act which was itself methodologically and epistemically apart. Between the second and third acts, then, brews a call for philosophical Euclideanism—one combining the methods and intuitions of Euclid's geometry. To derive from this a pedagogical mandate, however, required a justification at the nexus of the first and second acts. There, Cauchy's lionization of relations between real quantities should remind one that he was trained as an engineer, started his career applying his mathematics in the field, and taught at France's premier school for applied mathematics. And while those who draw inspiration from Cauchy's words today might think of themselves (and Cauchy) as pure mathematicians, the audience for whom Cauchy wrote his introduction—including students, administrators, and friendly and hostile colleagues—worked in a world where mathematics and engineering were practically and theoretically indissociable. The requirements of engineering did not in themselves drive Cauchy's mathematical moves, but neither can it be denied that Cauchy's engineering contexts shaped both what Cauchy valued in mathematical analysis and how he could go about inculcating those values to others.[56] Moreover, it cannot be a coincidence that Cauchy advocates for the separation of mathematics from other spheres of knowledge at a time when the École and his place in it were regularly threatened by the political machinations surrounding them.

Indisputably, Cauchy's work advanced the practice of mathematics, and even his introduction stresses the new questions opened up by his approach. But Cauchy's gains were won by cutting out a wide swath of the field. The disciplinary hygiene espoused in his introduction allowed Cauchy to rule out of bounds a large array of mathematical activity. Confronted with a new class of mathematically educated amateurs and aspirants educated in elite engineering schools such as the École Polytechnique,[57] Cauchy defended his discipline by policing its modes of right practice. Controlling methods was an integral part of controlling access.[58] By radically narrowing what could count as right method, Cauchy cut the ground from underneath a wide body of would-be mathematicians. Of course, Cauchy was not alone in this ambition, and the disciplinary contraction of mathematics took place at many levels. Polytechnicians of all stripes rallied around analysis as a challenging and rigorous method suitable for an elite and highly selective

[56]To be sure, the connection to engineering is oblique or occluded elsewhere in Cauchy's work, and even in other sections of the Course. The relative clarity of engineering's role in Cauchy's thought in his introduction is a mark of the particular audiences, as well as of the particular goals and strategies, of that portion of his text. Freudenthal (1971, p. 377) reminds us that the philosophical and mathematical relationship between Cauchy's Course and the rest of his work is anything but straightforward.

[57]See Alder (1999).

[58]Cf. Hunt's (1991) characterization of the opposition to Oliver Heaviside's mathematics.

school (Gillispie, 1994, p. 40).[59]

A product of the many contexts from which it was born, Cauchy's introduction was optimally placed both to reflect and to help shape a remarkable sea change in European mathematics. As a reflection of the movement it helped to define, the introduction offers a glimpse of the epistemic contortions underlying what on the surface was a simple assertion of the priority of rigorous mathematics. Taking seriously the web of allusions undergirding Cauchy's argument makes plain the conceptual alliances, both likely and unlikely, that made Cauchy's rigorous mathematics possible.

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Alder, K., 1995. A Revolution to Measure: The Political Economy of the Metric System in France. In: Wise, M.N. (Ed.), The Values of Precision. Princeton University Press, Princeton, pp. 39–71.

Alder, K., 1999. French Engineers Become Professionals; or, How Meritocracy Made Knowledge Objective. In: Clark, W., Golinski, J., Schaffer, S. (Eds.), The Sciences in Enlightened Europe. University of Chicago Press, Chicago, pp. 94–125.

Alexander, A., 2006. Tragic Mathematics: Romantic Narratives and the Refounding of Mathematics in the Early Nineteenth Century. Isis 97, 714–726.

Arnold, D.H., 1981. Poisson and Mechanics. In: Métivier, M., Costabel, P., Dugac, P. (Eds.), Siméon-Denis Poisson et la science de son temps. École Polytechnique, Palaiseau.

Belhoste, B., 1984–1985. Le Cours d'analyse de Cauchy à l'Ecole polytechnique en seconde année. Sciences et techniques en perspective 9, 101–178.

Belhoste, B., 1985. Cauchy, 1789–1857: un mathématicien légitimiste au XIXe siècle. Belin, Paris.

Belhoste, B., 1991. Augustin-Louis Cauchy: a biography, translated by F. Ragland. Springer, New York.

Belhoste, B., 1994. Un modèle à l'épreuve. L'École polytechnique de 1794 au Second Empire. In: Belhoste, B., Dahan Dalmedico, A., Picon, A. (Eds.), La formation polytechnicienne: 1794–1994. Dunod, Paris, pp. 9–30.

Belhoste, B., 2001. The École Polytechnique and Mathematics in Nineteenth-Century France. In: Bottazzini, U., Dahan Dalmedico, A. (Eds.), Changing Images in Mathematics: From the French Revolution to the New Millenium. Routledge, London, pp. 15–30.

Belhoste, B., 2003. La formation d'une technocratie: L'École polytechnique et ses élèves de la Révolution au Second Empire. Belin, Paris.

Benis-Sinaceur, H., 1973. Cauchy et Bolzano. Revue d'histoire des sciences 26(2), 97–112.

Bernard, P., 1911. Confession (du Concile de Latran au Concile de Trente). In Vacant, A., Mangenot E. (Eds.), Dictionnaire de Théologie Catholique, volume 3. Letouzey, Paris, cols. 894–926.

[59] See also Terrall (1999, p. 247). Foundational retrenchments such as Cauchy's are not uncommon in histories of restoration-era mathematics, whether the restoration is in seventeenth-century England (Shapin and Schaffer, 1985) or nineteenth-century Italy (Mazzotti, 1998). Rather than a constant increase in rigor and precision, the history of mathematics reflects the cyclical deployment of foundations and formalisms to suit the social, political, and mathematical needs of its practitioners—with Euclidean geometry an oft-traded token in historical controversies over mathematical rigor.

Bottazzini, U., 1992, Geometrical Rigour and 'Modern Analysis', an Introduction to Cauchy's Cours d'analyse. In: A.L. Cauchy, Cours d'analyse de l'École royale polytechnique, 1ère partie, Analyse algébrique. CLUEB, Bologna, pp. xi-clxvii.

Bottazzini, U., 2001. From Paris to Berlin: Contrasted Images of Nineteenth-Century Mathematics. In: Bottazzini, U., Dahan Dalmedico, A. (Eds.), Changing Images in Mathematics: From the French Revolution to the New Millenium. Routledge, London, pp. 31–47.

Boyer, C.B., 1968. A History of Mathematics. John Wiley & Sons, New York.

Bradley, R.E., Sandifer, C.E., 2009. Cauchy's Cours d'analyse: An Annotated Translation. Springer, New York.

Cauchy, A-L., 1821. Cours d'Analyse de l'École Royale Polytechnique; 1^{re} Partie. Analyse Algébrique. Debure, Paris. Reprint Éditions Jacques Gabay, Sceaux, 1989.

Cauchy, A-L., 1823. Résumé des Leçons données a l'École Royale Polytechnique, sur le Calcul Infinitésimal. Debure, Paris.

Chatzis, K., 1994. Mécanique rationnelle et mécanique des machines. In: Belhoste, B., Dahan Dalmedico, A., Picon, A. (Eds.), La formation polytechnicienne: 1794–1994. Dunod, Paris, pp. 95–108.

Costabel, P., 1981. Siméon-Denis Poisson: Aspect de l'homme et de son oeuvre. In: Métivier, M., Costabel, P., Dugac, P. (Eds.), Siméon-Denis Poisson et la science de son temps. École Polytechnique, Palaiseau, pp. 1–21.

Dahan Dalmedico, A., 1992. Mathématisations: Augustin-Louis Cauchy et l'École Française. A. Blanchard, Paris.

Dedekind, R., 1948. Essays on the Theory of Numbers. Open Court, La Salle (translated from the German text of 1901 by W.W. Beman).

Derrida, J., 1974. Of Grammatology, translated by G. C. Spivak. Johns Hopkins University Press, Baltimore.

Fox, R., 1974. The Rise and Fall of Laplacian Physics. Historical Studies in the Physical Sciences 4, 89–136.

Fraser, C.G., 1997. Calculus and Analytical Mechanics in the Age of Enlightenment. Variorum, Aldershot.

Freudenthal, H., 1971. Did Cauchy Plagiarize Bolzano? Archive for History of Exact Sciences 7(5), 375–392.

Gilain, C., 1989. Cauchy et le Cours d'Analyse de l'Ecole Polytechnique. Bulletin de la Société des amis de la Bibliothèque de l'École polytechnique 5, 3–145.

Gillispie, C.C., 1994. Un enseignement hégémonique: les mathématiques. In: Belhoste, B., Dahan Dalmedico, A., Picon, A. (Eds.), La formation polytechnicienne: 1794–1994. Dunod, Paris, pp. 31–44.

Gillispie, C.C., 1997. Pierre-Simon Laplace, 1749–1827: A Life in Exact Science, with R. Fox and I. Grattan-Guinness. Princeton University Press, Princeton.

Gillispie, C.C., 2004. Science and Polity in France: The Revolutionary and Napoleonic Years. Princeton University Press, Princeton.

Grabiner, J.V., 1981. The Origins of Cauchy's Rigorous Calculus. The MIT Press, Cambridge, MA.

Grabiner, J.V., 1997. Was Newton's Calculus a Dead End? The Continental Influence of Maclaurin's Treatise of Fluxions. The American Mathematical Monthly 104(5), 393–410.

Grabiner, J.V., 2004. Newton, Maclaurin, and the Authority of Mathematics. The American Mathematical Monthly 111(10), 841–852.

Grattan-Guinness, I., 1970. Bolzano, Cauchy and the 'new analysis' of the early nineteenth century. Archive for History of Exact Sciences 6(5), 372–400.

Grattan-Guinness, I., 1980. The emergence of mathematical analysis and its foundational progress, 1780–1880. In Grattan-Guinness, I. (Ed.), From the Calculus to Set Theory, 1630–1910. Duckworth, London.

Grattan-Guinness, I., 1990. Convolutions in French Mathematics, 1800-1840: from the calculus and mechanics to mathematical analysis and mathematical physics. Birkhäuser, Basel.

Grosholz, E.R., 1980. Descartes' Unification of Algebra and Geometry. In: Gaukroger, S. (Ed.), Descartes: Philosophy, Mathematics and Physics. Harvester Press Ltd., Sussex, pp. 156–168.

Hogben, L., 1960. Mathematics in the making. Macdonald, London.

Høyrup, J., 1996. The formation of a myth: Greek mathematics—our mathematics. In Goldstein, Gray, Ritter (Eds.), L'Europe mathématique / Mathematical Europe. Éditions de la Maison des sciences de l'homme, Paris, pp. 103–122.

Hunt, B.J., 1991. Rigorous Discipline: Oliver Heaviside Versus the Mathematicians. In: Dear, P. (Ed.) The Literary Structure of Scientific Argument: Historical Studies. University of Pennsylvania Press, Philadelphia, pp. 72–95.

Iacobacci, R.F., 1965. Augustin-Louis Cauchy and the Development of Mathematical Analysis. New York University School of Education doctoral dissertation.

Katz, V.J., 2009. A History of Mathematics: An Introduction, 3^{rd} edition. Addison-Wesley, Boston.

Lakatos, I., 1979. Proofs and Refutations: The Logic of Mathematical Discovery, edited by J. Worrall and E. Zahar. Cambridge University Press, Cambridge.

Laugwitz, D., 1987. Infinitely Small Quantities in Cauchy's Textbooks. Historia Mathematica 14, 258–274.

Mahoney, M.S., 1980. The Beginnings of Algebraic Thought in the Seventeenth Century. In: Gaukroger, S. (Ed.), Descartes: Philosophy, Mathematics and Physics. Harvester Press Ltd., Sussex, pp. 141–155.

Mazzotti, M., 1998. The Geometers of God: Mathematics and Reaction in the Kingdom of Naples. Isis 89, 674–701.

Montucla, J.F., 1802. Histoire des mathématiques, nouvelle édition, considérablemennt augmentée, et prolongée jusque vers l'époque actuelle. Volume III. Henri Agasse, Paris.

Ortolan, T., 1911. Confession. Questions Morales et Pratiques. In Vacant, A., Mangenot E. (Eds.), Dictionnaire de Théologie Catholique, volume 3. Letouzey, Paris, cols. 942–960.

Porter, T.M., 1995. Trust in Numbers: The Pursuit of Objectivity in Science and Public Life. Princeton University Press, Princeton.

Richards, J.L., 1991. Rigor and Clarity: Foundations of Mathematics in France and England, 1800–1840. Science in Context 4(2), 297–319.

Richards, J.L., 1992. God, Truth, and Mathematics in Nineteenth-Century England. In: Nye, M.J., Richards, J.L., Stuewer, R.H. (Eds.), The Invention of Physical Science: Intersections of Mathematics, Theology and Natural Philosophy Since the Seventeenth Century. Kluwer Academic Publishers, Dordrecht.

Richards, J.L., 2006. Historical Mathematics in the French Eighteenth Century. Isis 97(4), 700–713.

Rusnock, A., 1995. Quantification, Precision, and Accuracy: Determinations of Population in the Ancien Régime. In: Wise, M.N. (Ed.), The Values of Precision. Princeton University Press, Princeton, pp. 17–38.

Shapin, S., Schaffer, S., 1985. Leviathan and the Air-Pump: Hobbes, Boyle, and the Experimental Life. Princeton University Press, Princeton.

Terrall, M., 1999. Metaphysics, Mathematics, and the Gendering of Science in Eighteenth-Century France. In: Clark, W., Golinski, J., Schaffer, S. (Eds.), The Sciences in Enlightened Europe. University of Chicago Press, Chicago, pp. 246–271.