Slightly Scalene Mathematics

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1 Introduction: Proofs and Examples

Mathematics is a proving discipline. Mathematicians use proofs to generate and contest new knowledge and to organize established knowledge. Proofs figure centrally in mathematicians' operational vocabulary, disciplinary values, philosophy, education, and pedagogy, as well as in their discipline's rich history and lore.

Proofs' power lies in their ability to establish claims of absolute generality with absolute certainty. Drawing upon established results and accepted modes of reasoning, mathematicians manage to pronounce unequivocal truths about *all* triangles, *all* sets, or *all* of any mathematical object subject to any mathematical conditions.

Of course, the practice of proofs is somewhat messier than the foregoing would imply. Mathematicians can disagree about what results have been adequately established or what modes of reasoning are acceptable. They can identify and rectify errors and omissions in their claims and reasoning. The steps of a proof can be hard to follow or evaluate, and the proof as

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a whole can be even harder. And then there's the matter of where proofs come from and what makes them significant. Nevertheless, in spite of these obstacles, mathematicians manage to agree that a well-executed proof can say things about collections of mathematical objects or ideas which are typically innumerable, unrealizable, and, in many cases, unimaginable.

This paper considers an elementary proof from Euclidean geometry in order to explain, in part, how this transit from the local and the particular to the general and the arbitrary is achieved. The proposition claims that the sum of the interior angles of any triangle is equal to two right angles. Even this simple example takes us into the realm of the infinite and the inconceivable. How can we ever hope to draw, or even imagine, every possible triangle? Our simple proposition applies to triangles larger than our physical universe, smaller than an atom, wider than an ocean, and narrower than a needle. It makes claims of such numerical precision that no instrument, real or imagined, could ever verify them through measurement.

In the face of such stark absolutes and broad generalities, one might assume that particular concrete mathematical examples—the triangles on a blackboard, the lines on a page—would have at best a minor supporting role in mathematical proofs. Not so, argued Imre Lakatos in his influential *Proofs and Refutations*, published posthumously in 1976 and based on work first shared by the author more than a decade earlier. Lakatos presents a rational reconstruction of the history of a theorem about polyhedra,[1] manifesting a dialectic whereby successive proofs and refutations lead to refinements of the theorem in question and the definitions and principles on which it is based.

Examples play two crucial roles in Lakatos's account. First, they support the heuristic reasoning and imagination giving rise to the series of proofs of the theorem. Simple imagined polyhedra allow Lakatos's protagonists to navigate the impassable realm of full mathematical generality and furnish them with a vocabulary for describing their proofs to each other. Second, examples are centerpieces in the process of refutation. In the face of a supposedly general theorem about polyhedra, different putative polyhedra vie

[1]The theorem asserts that the number of vertices plus the number of faces minus the number of edges in a polyhedron always equals 2.

for the status of "counterexample." Depending on whether and how they are accepted as counterexamples or are deemed too monstrous to merit that distinction, these examples lead to the reshaping or outright rejection of proofs, propositions, and supporting definitions.

Lakatos thus presents us with a paradox: proofs and propositions deal in mathematical *generalities*, yet every stage of mathematical proof-making is shaped and constrained by *particular* examples.

2 Examples and Witnessing

This paper seeks to resolve Lakatos's paradox by interrogating what it is about particular examples that gives them license to speak for general phenomena. To do so, I would like to return to a well-trod literature in the history and sociology of science that has not received sufficient attention in corresponding studies of mathematics. That literature, whose principal touchstone is Shapin and Schaffer's (1985) *Leviathan and the Air-Pump*, concerns scientific witnessing.

This may seem like an odd choice on which to ground my study. Witnessing in science is a means of mastering unruly natural phenomena in a shared and controlled environment, whereas mathematicians deal in orderly abstract phenomena that seem from the outset to be universally accessible. As Shapin (1988) argues, the Boylean regime of scientific witnessing was established in explicit contradistinction to the methods and objects of mathematics.

Nevertheless, a number of similarities between mathematical proofs and scientific arguments suggest that witnessing may be just as essential to mathematics as it is to the natural sciences, even if it operates in different ways. Both mathematical and scientific demonstrations aim to convince a wide body of specialists of the validity of a locally produced fact. Scientists must convince colleagues that conclusions drawn from their specific laboratory apparatus apply both in other laboratories and in nature. Mathematicians, on the other hand, must win assent for their own mathematical insights and convictions from their peers, whether those insights are developed collaboratively or in comparative isolation.[2] Witnessing offers a social technology that can help both scientists and mathematicians to do so.

Both combine material and literary technologies to expand the reach of their results. Scientists build equipment, distribute samples, and attempt replications, but they also rely on plausible accounts given by other experimenters. Mathematicians seem at first to deal only in the ideal world of their objects of study, but ultimately rely just as much on written formalisms on blackboards, papers, and computer screens as do scientists on their laboratory equipment.[3] Just as scientists cannot reproduce—for practical reasons among many others—every experiment whose results they credit, so too are mathematicians unable to verify every step of every proof they believe to be valid. In particular, both scientists and mathematicians learn to assess whether a result "looks right" and to adapt their critical attentions accordingly. Both rely on a combination of trust, testimony, verisimilitude, and logical argument in order to certify new knowledge.

Where scientists use witnessing to establish matters of fact, mathematicians can be seen to use witnessing as a technology of proving to establish the truth of theorems. Treating methods of mathematical proof under the rubric of Boylean witnessing (as presented by Shapin and Schaffer) draws attention to two key features of proofs whose existence or relationship to one another might otherwise elude notice. First and foremost, the Boylean model stresses the importance of verisimilar settings and descriptions. For "virtual witnessing" to extend the reach of a scientific experiment, the conduct and reports of those experiments must be recognizable while also avoiding the appearance of having been manipulated to favor the experimenter's conclusions. In mathematics, examples must likewise be designed and chosen to be recognizable without appearing to favor the theorem's conclusion unduly. In both cases, the objects under consideration must be construed as pre-theoretical. Moreover, verisimilitude in both cases has a universalizing function, present-

[2]Practitioner accounts like Thurston (1994) and DeMillo, Lipton, and Perlis (1979) show that mathematicians themselves see this to be a weighty issue.

[3]On this, see especially Barany and MacKenzie (forthcoming), Rosental (2008), and the literature reviewed in both.

ing a locally achieved phenomenon as "just another" instance of something universal and general.[4]

Second, this style of demonstrating represents a *social* solution to an *epistemological* problem. That is, in both cases the demonstrator builds on carefully calibrated communal norms and expectations to bootstrap away an irresolvable difficulty inherent to their manner of demonstrating. In the natural sciences, the difficulty is the contrast between the local and artifactual nature of the laboratory against the vast plurality of nature. In mathematics, it is the contrast between the irreducible material specificity of the proof's exemplars against the abstract generality of the theorem's objects. The respective epistemological gaps are bridged with value-laden and norming discourses that rest on the social configurations of the respective disciplines.

The observation that the passage from the particular to the universal in mathematics is ultimately social is often associated with meaning-finitism and the Edinburgh school of the Sociology of Scientific Knowledge.[5] Contrasting the finite nature of our lived experience with the infinite potential of our future experience, meaning finitists argue that we constantly make and revise open-ended classifications based on relationships observed in what we experience. Bloor (e.g. 1973, 1976, 1978) has done the most to apply this principle to mathematical proof, treating (as I shall do below) examples from elementary mathematics alongside those from Lakatos's study. He argues that mathematical statements have no transcendent or universal meaning, and are instead pliable generalizations from the finite examples of mathematicians' experience. Which examples are salient and how generalizations are made, for Bloor, depend on social interests and negotiations.

This view was challenged and enriched in an exchange of essays on Wittgenstein and rule-following between Bloor (1992) and Lynch (1992a, 1992b). Both agreed that the way people produce general mathematical statements

[4]This last function is related to the "suspension of disbelief" discussed in Corry's (2007) article on poetic license in mathematical narrative.

[5]For a general introduction to this school of finitism, see Barnes (1982) and Barnes, Bloor, and Henry (1996).

was ultimately social and conventional, but they disagreed on the form those conventions took. Where Bloor presents mathematical understandings as subject to a large and often implicit web of social forces, Lynch places a greater emphasis on shared and learned practices of understanding and argumentation. Here, Lynch drew from an emerging literature in the ethnomethodology of mathematics whose chief expositor has been Eric Livingston (e.g. 1986, 1999, 2006). Livingston stresses the variety of discursive and practical resources used in mathematical arguments, showing how the choice of a particular example, method, or way of viewing a mathematical phenomenon affects the kinds of reasoning and inferences that can follow.

The core of this paper is modeled on Livingston's method of "demonstrative sociology" (2006, 64–68), in particular using elementary theorems of Euclidean geometry to elucidate broader features of mathematical practice. Livingston uses geometric diagrams and proofs to show how the adequacy and objectivity of mathematical reasoning is established as a "witnessible achievement" in a valid proof (1986, 5). According to Livingston, mathematicians' lived work of proving is essential to understanding the ways in which aspects of a proof become remarkable or accountable. Specific modes of presentation render certain figures as arbitrary and certain deductions as logical or necessary. Because of his emphasis on the "intimate details of proving theorems" (2006, 65), Livingston finds his strongest evidence in doing mathematics along with his readers while calling attention to the analytically significant features of what is being done.

Demonstrative sociology, in this way, relies on a hybrid of literary "close reading" and what might be called "close practicing" to elucidate how examples are deployed and rendered workable in proofs. Where Livingston uses these methods to address social coordination and mathematical inference, Rotman (e.g. 1988, 1997) turns them to the semiotic analysis of mathematical signification. Like Rotman, I shall ask what mathematical figures and expressions are taken to mean, and what features of those inscriptions facilitate that meaning—particularly where that meaning involves a leap from the particular to the universal.

Livingston's and Rotman's approaches complement the ethnographic method of observing and interviewing practicing mathematicians as they do their work. Below, I expand on the insights from my Livingstonian analysis with some observations from my recent ethnographic work on university mathematics researchers.[6] Of particular relevance among prior ethnographies of proof is Rosental's (2008) study of demonstration and validation in an online community of logicians. Rosental (ibid., 98, 101) introduces the term "demonstration," which in French encapsulates the embeddedness of showing (*montrer*) in proving (*démontrer*), and explains how actors mobilize around a series of shared formalisms (or "showings") in order to manufacture self-evident conclusions, and thus reach consensus concerning a logical argument.

The following analyses demonstrate how examples in mathematics permit social coordination in proofs by deploying a special sort of verisimilitude. Using what I shall call "slightly scalene" examples, mathematicians make use of a process remarkably similar to Boylean witnessing to make universal claims from local inscriptions.

3 A Slightly Scalene Example

In planar geometry, a scalene triangle is one with no two sides of the same length (figure 1). The triangles of our everyday experience—from textbooks to tiles to traffic signs—are typically equilateral, with all sides and angles the same. When small children first learn to distinguish between shapes, equilateral triangles figure prominently in that education. Such triangles (at least when moderately sized) are recognizable, reproducible, circulable, and easily diagnosable as having three sides joined by three angles.

Even though equilateral triangles would seem to be the most exemplary of triangles in the everyday sense of exemplarity, a mathematician would never use an equilateral triangle in the figure for a proof about arbitrary triangles. The reason has to do with a crucial difference between how mathematicians conceive of specificity and generality and what the terms imply in everyday

[6] More comprehensive treatments of that project and its theoretical underpinnings can be found in Barany (2010) and Barany and MacKenzie (forthcoming). See also Greiffenhagen (2008).

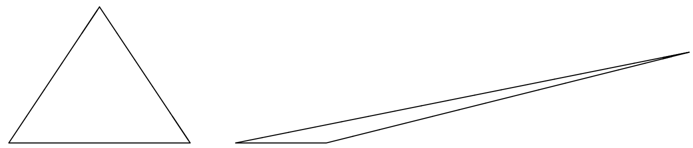

Figure 1: A non-scalene and a scalene triangle.

usage. Generality, in mathematics, is understood in terms of the range of things to which a formulation applies, or the range of things that could share an object's mathematically pertinent properties. Specificity entails the opposite criterion: the most specific mathematical formalisms are the ones that apply to the fewest objects, and the most specific objects are the ones sharing their pertinent properties with the fewest others.

All triangles share certain mathematically pertinent features. In Euclid's *Elements*, they are characterized by having exactly three distinct sides, and they share all properties that follow from that characterization (including that those sides form three distinct angles). Each triangle also has its own incidental features, including the particular measures of its sides and angles, its location in the plane, its position with respect to other objects, and the way in which its points, sides, and angles are labeled (if at all). All of these incidental features may play a role in a proof about a particular triangle, but none of these affects its being a triangle in good standing.

In addition to the pertinent properties they share with all triangles, equilateral triangles all have further salient properties in common. In particular, it is mathematically relevant that all their sides and angles have equal measure. The particular length of their sides may vary,[7] but, wherever situated and however large, an equilateral triangle will always have sides of equal length. This means that while every equilateral triangle shares the *mathematically pertinent* properties of scalene triangles, the converse is not the case. Likewise, propositions about equilateral triangles (for instance, the proposition that their internal angles are each 60 degrees) will not typically

[7]Their angles, however, have the same measure regardless of side-length.

also apply to scalene triangles (whose angles, for instance, can vary between 0 and 180 degrees).

In the mathematical sense, then, an equilateral triangle is more specific than a scalene one and a proposition about equilateral triangles is more specific than a proposition about arbitrary triangles. (Because "scaleneness" is generally not seen to be a pertinent feature of scalene triangles, one almost never finds propositions pertaining specifically to scalene triangles that do not also apply to ones that are equilateral or isosceles—with two of three sides and angles being equal.) Consequently, while equilateral triangles are the most general in the everyday sense, scalene triangles are the most mathematically general.

This generality is important in mathematical reasoning because more general objects share the most pertinent features with *arbitrary* representatives of the class of objects under consideration. In a theorem about triangles, one wants to reason with an exemplary triangle whose pertinent features are present in every single object—concrete, imagined, or unimaginable—to which the theorem is to apply. If one were to reason instead with an equilateral triangle, it would take a special effort to avoid conclusions which rely inextricably on its special shape.[8] Accordingly, for a triangle to be used successfully in a proof it must be distorted enough from an equilateral one that the prover can claim that all of its properties derive from the general structure of ideal triangles and that the proof uses only essential features shared by all triangles.

Of course, there are many ways to distort a triangle, and no set of criteria will designate one particular triangle as more general than all the rest. Moreover, the exemplary triangle must remain workable. Triangles larger than our physical universe or smaller than atoms, for instance, need not apply. Lastly, an exemplary triangle must not be so distorted that it is no longer recognizable as a triangle. If a triangle has so narrow a base that its picture on a page is indistinguishable from the picture of a line, it fails not just to be

[8] Of course, such "example traps" can still occur with scalene triangles. Livingston (2006, 40) gives one such example of reasoning with a scalene triangle that does not lead to a successful proof.

workable but also to perform as a triangle. A successfully exemplary triangle must be enough like an equilateral triangle to be workable and exemplary, but enough unlike an equilateral triangle to be fruitful and general. It must be scalene, but only slightly.

As we shall see, the notion of a slightly scalene example goes well beyond triangles and geometry. All examples are characterized by their particularity, instantiating mathematical phenomena in a specific and local way. Successful examples are intuitive, recognizable, circulable, and manipulable in ways only possible for a narrow collection of potential examples. On the other hand, examples must function within a mathematics of the universal. A successful example is suggestive of general phenomena, and leads provers to exploit in their reasoning those of its features which are indicative of all objects represented by the example. To see how this extends beyond the simple case of triangles, consider the following proof, adapted from Proposition 32 of Euclid's *Elements*.[9]

Proposition. The internal angles of a triangle sum to two right angles.
Proof.

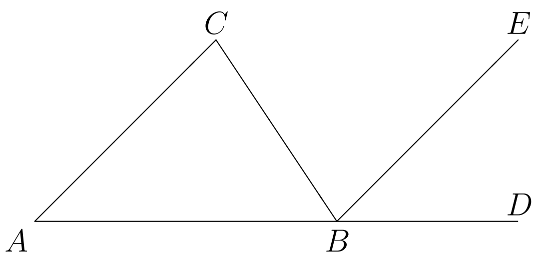

Figure 2: Construction for summing angles of a triangle.

Let $\triangle ABC$ be an arbitrary triangle (figure 2). Apply Euclid's postulate I.2[10] to extend the line segment AB to some point D. According to Euclid's Proposition I.31,[11] construct a line segment BE through the point B and

[9] Heath (1956, 316–322) offers a definitive English translation and commentary.

[10] Any finite straight line can be continuously extended to a straight line of any length.

[11] Given a line and a point, another line can be constructed through the point which is parallel to the original line.

parallel to the segment AC. By Proposition I.29,[12] since AC is parallel to BE we have

$$\angle CAB = \angle EBD \qquad \text{and} \qquad \angle ACB = \angle CBE.$$

Now, $\angle CBA$ is complementary to the sum of the angles $\angle CBE$ and $\angle EBD$ along the line ABD, so the sum of those three angles is equal to two right angles (I.13).[13] Thus, the sum of the angles of the triangle will also equal two right angles. This completes the proof.

There is one obvious slightly scalene example used in this proof: the triangle in figure 2. It is clearly recognizable as a triangle. We can see all three of its angles and all three of its sides in a single glance. We can also readily see that each of its sides and angles has a different magnitude. AB is clearly the longest side, followed by AC and then BC. Likewise, the angle at vertex A is the sharpest, the angle at vertex C is slightly less than a right angle, and the angle at vertex B is somewhere in between.

The argument of the proof consists of reproducing the three angles of the triangle at the single point B and then observing that the angles, thus arrayed, form a straight line. A meticulous prover would verify each step of the reproduction, including those in the cited propositions. But the example works well for this proof in part because it is already easy to see that the three angles assembled at vertex B are the same as those of the triangle. This ease comes from the relative sizes of the three angles and the diagram's near-preservation of the orientation of each angle, creating multiple non-verbal markers of angle identity to supplement and reinforce the use of labels to mark angles in the proof's text. (For the angle at vertex C, the corresponding angle undergoes a half rotation.) The triangle from the figure successfully dramatizes the crucial general fact underlying the proof's technique, that all triangles consist of three distinct but comparable angles that can be rear-

[12]When a line crosses two parallel lines, the alternate angles it forms are equal to each other. The proposition is applied twice: once with AD as the crossing line and once with CB as the crossing line.

[13]Complementary angles sum to two right angles.

ranged in other ways. For nearly every reader, these non-essential features of the triangle in the figure do far more to assure the credibility of the proof and the propositions it cites than do the proof's systematic deductions and citations themselves.

But the triangle in figure 2 is not the only slightly scalene example at play. The system of labels in the figure and the corresponding nomenclature of the proof are also slightly scalene. One conceit of geometric figures is that it should matter neither how they are drawn nor how they are labeled—these are incidental properties of the figures. But some sets of labels are clearly superior to others for the instrumental purposes of the proof. They must, for one, function as practical points of reference in the proof's text. Unreadable, unpronounceable, or overly cumbersome labels would be too scalene. Moreover, there is no requirement that labels for related objects in the figure be similarly related in any systematic way. Yet the vertex labels in this proof form an alphabetic series corresponding to a counter-clockwise traverse of the triangle starting on the left.[14] The alphabetic series is particularly significant because such series are also used to denote algebraic variables in mathematical proofs, and so inherit the connotation of arbitrariness even when used to mark fixed locations in the diagram.

Angles and lines are denoted using a consistent system of reference to vertices, reinforcing the geometric relationships between points, lines, and angles in the figure. For both angles and lines, the labels also refer to the traverse of a hand gesture that might be used to indicate the object as well. For instance, one would indicate the segment AB by sweeping one's hand from vertex A to vertex B, and likewise the angle $\angle ACB$ by sweeping one's hand from A to C to B. Though these labels would be mathematically equivalent to separate referents involving proper names like "Eric," ideograms or barcodes, or systematic library-style classmarks, none of these systems embeds the kind of conventional recognizability and conceptual legibility that

[14]Vertex labels in geometric diagrams are almost always given as an alphabetic series. On the relationship between the order of vertex labels in the series and the proof's reference to those labels, see Netz's (1999) definitive account of the role of diagrams in classical Greek geometry. On the specific case of classical geometric diagrams containing more information than is used in the proof, see Saito (2009, 820).

the alphabetic system imparts.[15]

Labels can also be too equilateral. One would never label the angles of a diagram such as figure 2 with their measure in degrees, nor vertices with their coordinates on the page or in an abstract plane. Such a labeling gives too strong a reminder of the local particularity of the figure, emphasizing properties like location and measure that, with respect to the proof, are meant to be incidental. Nor should a label undermine or trivialize the result at hand. Although it would be mathematically correct to give both angles $\angle ACB$ and $\angle CBE$ the common label α, for instance (see figure 3), the labeling's facile conformity with the outcome of the theorem (and in particular the step of the proof asserting $\angle ACB = \angle CBE$)casts doubt on its legitimacy as a general labeling. The same is true of labels in terms of angles' measures. The proof that the angles sum to 180°, even if logically correct, loses its appeal to the arbitrary realm of all triangles when the method specific to the exemplary triangle of simply adding its angle measures becomes available. Think of someone asserting that all books are 315 pages long by handing you a book with the numeric label 315 on the last page—even if the statement were true this would be an implausible demonstration of it.

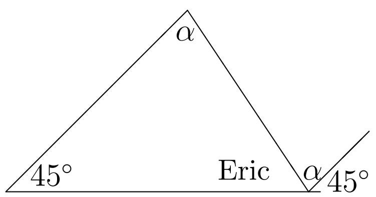

Figure 3: A construction and labeling for the slightly scalene triangle from the proof, with some overly-scalene or overly-equilateral features.

As the case of labeling would suggest, the principle of slightly scalene examples remains relevant even when there are no mathematically salient features to be rendered equilateral or scalene. Any orientation of figure 2 would

[15]To be clear, the alphabetic system works this way due to a combination of historical and cultural influences as well as material features of alphabetic inscriptions such as their size and optical distinguishability. I do not claim that the alphabetic system is inevitably the most slightly scalene system of labels.

be mathematically equivalent because the figure's orientation is incidental to the proof. No special distortions are necessary to make the particular orientation of the figure seem less facile. Nevertheless, some orientations are more legible than others, more in line with the "slightly" part of slightly scalene. Here, the triangle sits with its longest edge horizontal on the page and the extension of that edge to the point D is done in the left-to-right direction of reading used in the proof's text. Both of these conventional features of the diagram and proof make the argument recognizable to readers of this genre of geometric proving.

As a final example, contrast figure 3 with figure 2 from the proof. Figure 3, while mathematically correct in its depiction of the angles of the proof, renders the two angles outside of the triangle in such a way as to obscure their relationship to the angles of the triangle itself. In figure 2, to the contrary, the two auxiliary line segments BD and BE are drawn with a moderate length which increases the surveyability of the diagram, and hence of the argument as a whole. In particular, the visual similarity in size and orientation between the auxiliary segment BE and its counterpart AC from the triangle reinforces the mathematical relationship between them established in the proof. The ratio of their length, which is mathematically incidental, greatly increases the diagram's effectiveness at conveying the equality of the angles they produce, an equality that is established in the proof but should not in principle rely on the diagram for its plausibility. In the case of AB and BD, extending the segment BD far enough to establish visually that it is a continuation of AB has a similar effect.

In each of the ways just discussed, the diagram is produced so as to balance the mathematical and common notions of exemplarity, and so to appear at once arbitrary and recognizable. The effect of these slightly scalene representations is to create a verisimilar image that is familiar enough to be plausible without being so familiar as to compromise the proof's claim to generality. Like the Boylean experimental space, the proof's example reinforces conclusions both by contriving a controlled and comfortable setting for making its claims and by introducing enough gritty disorder to make the demonstration believable.

That disorder, as we learn from Shapin and Schaffer's account of Boyle's narratives, is indispensable. By making examples slightly scalene instead of equilateral, the prover disavows any special advantage a more orderly figure might convey and makes a show of proving with a less-than-ideal object in order to show that any object could take its place, even objects that, due to material or other constraints, manifestly could not do so. Thus, due to its particular form of particularity, the prover's example manages to implicate the entire field of objects to which the proof is said to apply.

Just as in Boylean witnessing, slightly scalene accounts implement a value-laden social solution to the problem of mathematical generality. They invoke a tripartite correspondence between the proof's deductions, its examples, and the abstract world of objects under consideration. As I shall show in my elaboration on Lakatos, below, this carries with it a contract for how proofs are to be validated or refuted. Namely, it is through the construction of counter-examples that one may definitively refute a claim (challenging the adequacy of the prover's scalene-ness), and it is by verifying the correspondence between the deductions and the example at hand that a proof is to be understood and rationalized (confirming the adequacy of the prover's slightly-ness). In either case, provers are invested with the trust that they have considered all possible factors that might discredit their assessment of an example's slightly scalene manifestation, even when the full range of examples far exceeds what it is possible to consider.

This conclusion adds depth to Livingston's claim that questions about a particular demonstration are "addressed by reference to the objects and to the proof that are being described[,] as part of the witnessed production of the mathematical demonstration" (1999, 873). When mathematicians contest claims in front of each other, Livingston observes, they do so with the aid of shared workable objects of reasoning. But that orientation toward the specific adequacy of a given representation in the proof extends to proof verification by individuals as well. In both situations, a prover's claims are assessed with reference to the proof's exemplary cases as if the demonstration were being witnessed by a group of mathematicians to which the proof's objects are simultaneously accountable. We have seen in the context of the above proposition that the kinds of considerations available in this context may be significantly circumscribed by the practical conventions of slightly scalene mathematics.

The social requirements for what counts as slightly scalene, combined with pedagogical and textual traditions, help to account for the remarkable stability of particular kinds of diagrams in Euclidean and other forms of mathematical argument. Figure 4 shows the diagrams corresponding to figure 2 as they appear in, respectively, the definitive sixteenth- and twentieth-century English translations of Euclid's *Elements* (Billingsley 1570, Heath 1956) and a recent and similarly definitive Internet translation using the Java programming language (Joyce 1996). All three use the same counter-clockwise ordering of the vertex labels for the proof's triangle (differing slightly from figure 2), starting at the lexically natural point closest to the top-left of the figure. In all three cases the segment CE exactly mirrors BA, suggesting just how crucial it is for those segments to be scannably equated for the proof to make sense.

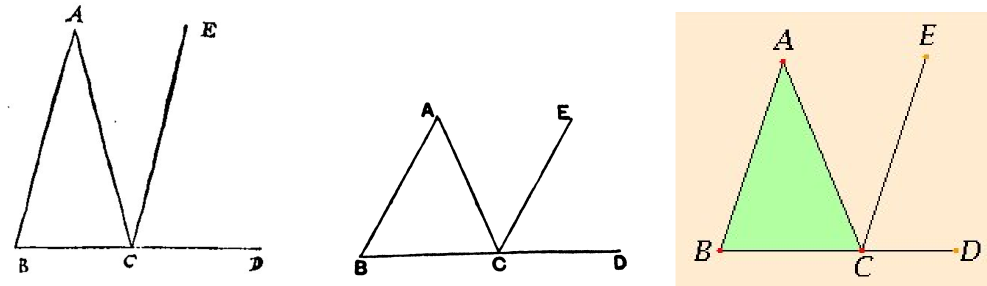

Figure 4: Three historical diagrams for proofs of Euclid's proposition I.32. Respectively: Billingsley (1570), Heath (1954), Joyce (1996).

Billingsley's triangle appears to be isosceles rather than scalene, while Heath's leans to the right and Joyce's to the left. The difference between the sixteenth- and twentieth-century figures here represents a shift in what counts as sufficiently scalene for this particular proof.[16] In no case is the triangle equilateral, nor in any case does it deviate far from being so.

[16]The historical reasons for this shift, which is even more pronounced in the case of the Pythagorean theorem in Early Modern transcriptions and translations of the *Elements*, are the subject of ongoing study.

The chief difference between Heath's and Joyce's triangles is not immediately apparent, but represents an important dependence for slightly scalene representations on the technologies of representation that are available. The advent of the Internet and personal computing between Heath's translation and Joyce's dramatically increased the possibilities for interactively customizing geometric diagrams. Clicking on the colored vertices in Joyce's figure allows the user to modify the image by repositioning those vertices subject to the constraints of the screen's resolution and field of vision. Joyce programmed the display so that if one vertex moved then every vertex whose position depended on that first vertex would also move in order to make the figure continue to match the proposition's written proof. Joyce's diagrams thus enact in yet another way the supposedly arbitrary nature of the exemplary triangle selected while further obscuring the extremely limited extent of such triangles available for producing intelligible proofs.

Yet even within the constraints of Joyce's manipulable image it is possible to exceed the limits of what is legible and workable in a proof. As figure 5 shows, some distortions can easily make Joyce's diagram functionally illegible. These modified diagrams collapse key but distinct features of the diagram onto one another and cast other features beyond the field of view. According to the terms of the proposition's deduction they are valid representatives, but it is simply not possible to follow or assent to the proof with any of these images as its operational referent, nor is it possible to verify the diagrams' validity by inspection. Rather than read this as a weakness of Joyce's translation, I would consider it a strength. These diagrams show that every aspiration to generality in mathematical proof—every effort to produce adequately scalene exemplars to justify one's argument—is subject to comparatively narrow limits on what works in practice.

4 Examples in Mathematics

Slightly scalene examples can be found in all areas of mathematics, not just geometry. Take the case of mathematical analysis, which studies the limiting

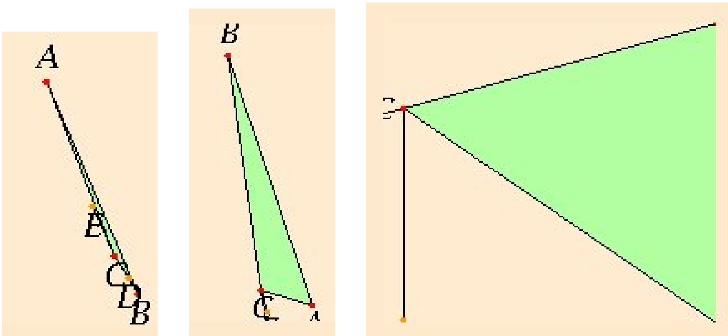

Figure 5: Three user-manipulated diagrams from Joyce's (1996) Java translation of the *Elements*.

properties of functions. By design, the objects of mathematical analysis are typically either unboundedly large or infinitesimally small. The mathematically interesting behaviors of individual functions are rarely manifest at any one fixed scale or view, much less those of the infinite collections of functions treated in analytic proofs. Nevertheless, mathematical analysts use graphs and other representations of functions alongside symbolic manipulations to develop intuitions and to make plausible demonstrations.

As students learn to do analysis, they acquire an inventory of slightly scalene functions adapted to particular cases and kinds of problems. They learn, for instance, that inferences about continuous functions of one variable can often be made soundly by reasoning with a smooth, closely bounded curve with few local maxima and minima and x-values not too far from zero. Some aspects of this example, such as its scale and location on the x-axis, can be justified after the fact as mathematically incidental. Others, such as the curve's shape and bounds, are there to keep the example from being so scalene as to be unworkable without necessarily having a clear justification.

I suggested above that nomenclature and methods of representing formulae have a slightly scalene character, and this observation was amply supported in my ethnographic investigations of university researchers in analysis. As in Euclidean geometry, alphabetic or other series of labels convey an impression of arbitrariness while giving a proof additional symbolic structure and legibility. Where it is important to differentiate between many kinds of formal objects in analysis, such as functions, covectors, points, and spaces, analysts use further typographic differentiation through changes of font, capitalization, language (switching in some cases to Greek letters), or starting point for the alphabetic series. In this last case, for instance, functions may be f, g, h while points might be x, y, z and scalars a, b, c, and it does not matter that the proof may invoke far more scalars than there are letters available (especially letters before f) because only a small collection of them are needed in the formal exegesis.

Figure 6 contains two images from the weekly research seminar where the analysts I studied shared their latest findings and hypotheses with each other. The left image is from a speaker's notes, and depicts a cone $\Gamma(x)$ (represented by a wedge) extending from the boundary of a domain (represented by the wavy line under the wedge). Here, the waviness of the line representing the domain's boundary is a scalene feature of the diagram, showing that the boundary could take any of a large number of smooth shapes but that it need not be free of curvature or other potentially pertinent properties. The wedge on the boundary-line is a standard cross-sectional way of representing cones in mathematics (Γ is likewise a standard notation for a cone), and the wedge here is moderately proportioned so as to be easily observable as a cone touching the boundary. The fact that the cone lacks a sharp point and spills over the boundary line in this figure is not understood to be salient and the figure is used as if it were more precisely drawn. Crucially, thanks to the flexibility of representational conventions, it need not be so precisely drawn to be used in this way.

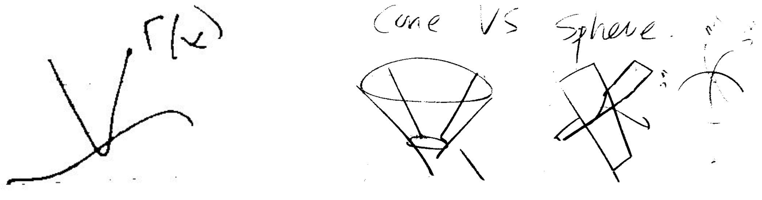

Figure 6: Slightly scalene figures from the analysis seminar.

The image on the right is taken from the blackboard writing of another speaker's talk.[17] It depicts a different representation of a cone, as well as a sphere, adapted to the concerns of the speaker's own argument. Here, what is relevant about the cone and the sphere is not how they relate to a boundary but how they intersect with other shapes. The cone is shown sliced near but not at its bottom vertex to illustrate the situation that can occur in that part of the object. Yet the intersection is drawn large enough to be rapidly diagnosable as a small circle, one whose size depends on its proximity to the bottom vertex of the cone. It is drawn this way even though the mathematically distinctive features of intersections with cones occur only as those intersections nearest to the limiting vertex. The illustration of the sphere shows that the kind of verisimilitude of slightly scalene examples need only apply to the particular mathematically pertinent features of the object under consideration at the time of the demonstration. Here, the depiction of the sphere looks like a slightly curved section of a plane rather than a round ball, conforming to its analytically relevant features at a small scale where the overall shape of the sphere is not detectable.

Both of these cases join with the historical examples above to illustrate further the dependence of slightly scalene representations on the media with which they are inscribed. Here, neither quick writing with a ball-point pen nor marking a blackboard with chalk permit the prover to create sharp, detailed images. For the depictions to be workable and legible, they must exist with a scale and shape that allows them to be rapidly and clearly produced as well as diagnosed.

The above analysis would suggest that there is a range (albeit a narrow one) of examples that can serve as slightly scalene tools in a proof. Not every mathematical situation, however, admits a slightly scalene exemplar. There is thus a genuine tension between "just scalene enough" and "not too scalene" that delimits what will work in a proof. The slightly scalene criterion is thus not only an explanation for how certain examples are effective in proofs, but also a test of which proofs can succeed in a given social context. History is laden with proof-arguments where a just-scalene-enough exemplar

[17] The colors have been digitally inverted and the image sharpened so as to be legible on paper.

is nevertheless too scalene or a workable exemplar is not scalene enough. So too is the historically-inspired progression of examples, counter-examples, monsters, and exceptions in Lakatos's *Proofs and Refutations* (figure 7).

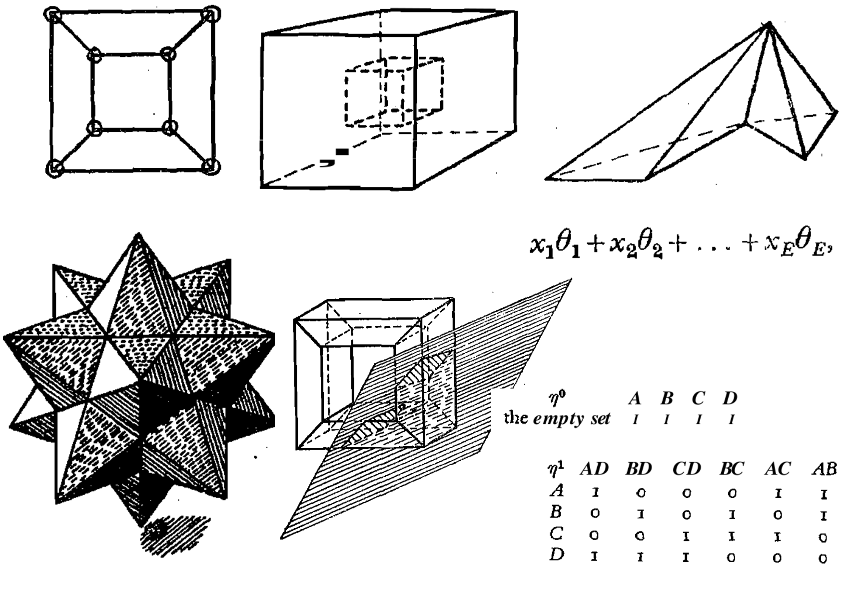

Figure 7: Images from Lakatos's *Proofs and Refutations.*

Lakatos's account shows how successive examples are judged to fail either in their exemplarity or in their generality. In the first case (as in the top-right image of the figure), they are too scalene and are thus excluded as monsters, where in the second (as in the top-left) they are insufficiently scalene and thus support theorems for which new counterexamples are later admitted. For Lakatos, the very truth or falsity of a theorem depends on which examples are admitted as truly exemplary, a determination that hinges on competing assessments of their slightly scalene-ness.

One might hesitate to say the same for theorems about triangles: an extremely scalene triangle might endanger one's ability to execute a proof, but it should not affect the underlying truth of the theorem. But it is precisely the corresponding reasoning with respect to polyhedra that Lakatos rejects.

Indeed, Livingston (2006, 40) offers a case where a possible lemma about triangles fails to generalize because it was based on an insufficiently scalene example. In Livingston's case, the lemma is thrown out in favor of a more scalene counterexample, but there is no intrinsic reason why that should have been the outcome. That one has an easier time distinguishing genuine examples from monsters for triangles than for polyhedra would seem, in our framework, more a difference in degree than a difference in the epistemic principle at play.

Moreover, as Lakatos amply demonstrates, it is often examples themselves that figure the important properties of a general mathematical object. Thus, the slightly scalene example is a proxy for a class of objects which it itself helps to delimit. One may say that a triangle is a figure in the Euclidean plane with exactly three edges, but all one can ever mean by this is that a triangle "looks like" one of a necessarily limited class of established exemplars for which this description is plausible.[18] This explains why candidate counterexamples often appear teratological and their users have to fight for their legitimacy. The slightly scalene concept thus aptly describes the sense in which, for a *valid example*, the words *valid* and *example* are always in competition. The more an example resembles the archetypal figures which lend it exemplarity, the less it is able to be used to suggest new global, abstract, or general properties for use in mathematical arguments. The more scalene (and hence valid for treating mathematical generalities) an example is made, the more it is liable to appear as a monster or to fail to appear at all.

In this way, entire mathematical theories develop through the deployment and refinement of the slightly scalene. Lakatos's rational reconstruction of the history of Euler's theorem is powerful because it distills the rules of disputation implicit in the slightly scalene regime of mathematical witnessing. Case studies and problems undergo successive formalizations, re-

[18]This claim is sympathetic with, but not reducible to, the strict meaning-finitist interpretation of classification. Specifically, as the examples from the beginning of this section show, what it means for an object to "look like" an exemplar in mathematics is considerably broader than is the case for the ostended objects invoked by meaning finitism.

presentations, and challenges as the specific is shaped into the general. For each of the polyhedra in figure 7 whose proponents claim them to be valid exemplars, the criteria for what counts as a slightly scalene polyhedron are challenged or reshaped in tandem with an evolving mathematical theory, just as those same polyhedra drive and embody the mathematics under dispute.

By the end of *Proofs and Refutations*, the slightly scalene examples are no longer images of polyhedra but exemplary algebraic equations. Only in algebraic form could Lakatos's imaginary students achieve the level of generality they sought from their representative polyhedra. Just as the algebraic equations appear to settle the matter, however, Lakatos throws a wrench in the works. Yes, the algebraisms appear to produce an unassailable proof, but their claim to exemplarity can never be so secure—formal equations take social and material work to be accorded the status of valid polyhedra just as much as the foregoing examples did. Lakatos's cascade of increasingly scalene polyhedra culminates in an algebraic object so scalene that its status as polyhedron is in doubt. Formalisms may produce a logically sound proof, but the determination of that to which the proof applies is irreducibly social. The particular and the "slightly" reassert themselves just as the march of mathematical abstraction appears most decisive. The story of Lakatos's polyhedra ends, in effect, by stressing the essential tension between particularity and universality all mathematical arguments must navigate.

5 Conclusion: Cascades of Inscriptions

In one of the most influential essays ever written on scientific representation, Bruno Latour identifies a "strange anthropological puzzle" at the heart of "the power of inscription" (1990, 52). In his view, the key to understanding how knowledge is made mobile, how it manages to transcend its local sites of production, is bound up with the problem of accounting for the training one receives (particularly in mathematics) "to manipulate written inscriptions, to array them in cascades, and to believe the last one on the series more than any evidence to the contrary." Here, Latour's attempt to locate the power of writing is both provocative and misguided. It is provocative because it finds in the apparent certainty and universality of mathematical formalisms the locus of certainty in science writ large. It is misguided because, as Lakatos so strongly suggests, mathematical conviction is never so strong as it seems. That conviction is always conditioned on certain limited forms of thinking through examples—examples whose adequate exemplarity can always be called into question.

And yet, mathematicians can and do call up the conviction to believe the outcome of their cascading inscriptions with a certainty for which philosophers and anthropologists alike have struggled to account definitively. It is a revisable certainty, but it is a certainty nonetheless. Evidence to the contrary can shake one's certainty in a particular proof, but for mathematicians the power of proofs is almost never in doubt. Certainly, it is not possible to doubt the power of proofs in one's capacity *as a mathematician.*

Mathematicians, Livingston (among others) has suggested, accomplish this feat of certainty by "disengag[ing] the mathematical object ... from the situated work that gives it its naturally accountable properties" (1986, 10). That is, the success of a mathematical argument is conditioned on its ability to sever its objects from its methods. I have argued here that the centrality of slightly scalene examples is an important consequence of Livingston's observation. Slightly scalene examples are workable and naturally accountable, but they are presented in such a way that they could be any object. This presentation succeeds, despite its manifest absurdity, through the social negotiation of what counts as sufficiently unprivileged in a mathematical argument. Like the Boylean experiment, the mathematical proof constantly teeters between unconvincing artifice and unworkable disorder.

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