Integration by Parts:

Wordplay, Abuses of Language, and Modern Mathematical Theory on the Move

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The following abbreviations and acronyms are used: AHES, *Archive for History of Exact Sciences*; AMM, *The American Mathematical Monthly*; BPL, Michael J. Barany, Anne-Sandrine Paumier, and Jesper Lützen, "From Nancy to Copenhagen to the World: The internationalization of Laurent Schwartz and his theory of distributions," *HM* 44, no. 4 (2017): 367-394; DPM, Michael J. Barany, *Distributions in Postwar Mathematics* (Ph.D. dissertation, Princeton University, 2016); HM, *Historia Mathematica*; LPD, Jesper Lützen, *The Prehistory of the Theory of Distributions* (New York: Springer, 1982); MGC, Laurent Schwartz, *A Mathematician Grappling with His Century*, tr. Leila Schneps (Basel: Birkhäuser, 2001); MI, *The Mathematical Intelligencer* 26, no. 1 (2004); PLS, Anne-Sandrine Paumier, *Laurent Schwartz (1915-2002) et la vie collective des mathématiques* (Ph.D. dissertation, Université Pierre et Marie Curie, 2014).

References to the MathSciNet online database of entries in *Mathematical Reviews* (http://www.ams.org/mathscinet/) or the zbMATH online database of entries in the *Zentralblatt für Mathematik* (http://zbmath.org/) use the standard numerical reference conventions for the respective databases, e.g. "MR0035918 (12, 31d)" and "Zbl 37.07301" for the first volume of Laurent Schwartz's textbook on the theory of distributions, which was the 35918th overall review in the MathSciNet database and was the fourth ('d') entry on page 31 of volume 12 of Mathematical Reviews, as well as the first entry on page 73 of volume 37 of the *Zentralblatt für Mathematik*.

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Abstract

This paper accounts for the intercontinental elaboration of French mathematician Laurent Schwartz's theory of distributions in the years immediately following World War II by tracing how mathematicians explained the theory to each other, advanced new interpretations, and reconciled existing ones. Situating distributions in mathematicians' changing contexts of funding, travel, and publication, especially in connection with the postwar reconstruction of international science, I argue that wordplay and suggestive comparisons—often termed 'abuses of language'—helped tie communities of scholars together across disparate geographies and fields of study. Material limits and linguistic ambiguity, here, offered important resources for asserting relevance and unity in a fragmented and heterogeneous discipline. I show in particular how reinterpretations and puns of the calculus technique of integration by parts helped advocates of Schwartz's theory create a far-reaching community of students and researchers that was itself partially integrated—with distributions' scholars believing themselves to be using a common theory while understanding and using that theory in considerably different (if sometimes mutually recognizable) ways. If exponents of modern mathematical research and pedagogy tend to emphasize settled theories and stabilized innovations, the history of these activities demands a converse emphasis on the variable and ongoing labor required to reconcile techniques and concepts, a labor that often hinges on theories' instability, pliability, and susceptibility to play.

Key words: mathematics, postwar, globalization, theory of distributions, Laurent Schwartz, wordplay, heterogeneous theory.

Short title: Integration by Parts.

[FIRST LEVEL HEADING] THEORY ON THE MOVE

No theory is born universal, and even the most monumental breakthroughs start small. In the winter months of 1944-1945, as the Second World War waned in Europe, a young French mathematician named Laurent Schwartz set out to rewrite the basic principles of mathematical analysis with what he soon termed his "theory of distributions."[1] In July, 1946, Schwartz lectured for four hours on his "great progress" with the theory to an approving audience of his countrymen, an eccentric collective of radical reformers who published under the pseudonym Nicolas Bourbaki.[2] A year later, Schwartz confided to his wife his unease with the "excessive success" his theory had found in Copenhagen, Denmark: "all the same, [the theory] is not Jesus Christ, and the compliments of the three Magi from the world over unsettle me a little; wasn't he later crucified?"[3]

In 1949, Schwartz took his first ever airplane trip on his way to lecture on his theory in Vancouver, Canada. The next summer, he received a prestigious Fields Medal at the International Congress of Mathematicians in Cambridge, Massachusetts, where his theory was compared to "Descartes' development of the analytic geometry" and his earliest paper on distributions was hailed as "one of the classical mathematical papers of our times."[4] In 1954, an elder statesman of Argentine mathematics all but rolled his eyes at the "enthusiasts of Lorenzo Schwartz's theory," a group that included many of his continent's leading young mathematicians.[5] By the end his theory's first decade, Schwartz had traveled tens of thousands of miles, crisscrossing continents, and his theory had traveled even farther. Scholars studied and taught variations of the theory in Europe, the Americas, Asia and Africa, across the Western and Eastern Blocs and in many parts of the non-aligned and developing world. Schwartz went on to be a leading figure in French and international mathematics, and distributions became a foundational element of postwar scholarship in several broad areas of mathematics and theoretical physics.

This kind of itinerary, for a young and previously little-known mathematician or his newborn theory, would have been unthinkable before 1945. The speed and, especially, the intercontinental geographical span over which mathematicians learned of and took to the theory defy comparisons to prewar precedents in mathematics.[6] After the Second World War, however, mathematicians and their sponsors reshaped the discipline's institutions, practices, problems and pedagogy. Stories like those of Schwartz and distributions became extreme examples of a normal pattern, rather than startling exceptions. Mathematicians pursued

[1] Schwartz's own retrospective narrative of his theory's origins is in MGC, ch. 6. For a critical appraisal and reconstruction of the first half-decade of this story, see BPL.
[2] Cartan to Weil, 19 Jul 1946, in Michèle Audin, ed., *Correspondance entre Henri Cartan et André Weil (1928-1991)* (Paris: Société Mathématique de France, 2011), 118. On Bourbaki, see Liliane Beaulieu, *Bourbaki: Une histoire du groupe de mathématiciens français et de ses travaux (1934-1944)* (Ph.D. dissertation, Université de Montréal, 1989); Maurice Mashaal, *Bourbaki: Une société secrète de mathématiciens* (Paris: Éditions Pour la Science, 2002).
[3] Laurent to Marie-Hélène Schwartz, 5 Nov 1947, Schwartz family archives, reproduced in Claudine Schwartz, "Autor des premiers travaux de Laurent Schwartz sur les distributions," *Gazette des mathématiciens* 113 (2007): 112-118, on 117-118. See also PLS, 134.
[4] Harald Bohr, "Address of Professor Harald Bohr," in Graves et al., eds., *Proceedings of the International Congress of Mathematicians, Cambridge, Massachusetts, U.S.A. 1950* (Providence: American Mathematical Society, 1952), 127-134, on 130, 133.
[5] Julio Rey Pastor, "La matemática moderna en Latino América," in *Segundo Symposium sobre Algunos problemas matemáticos que se están estudiando en Latino America, Villavicencio-Mendoza 21-25 julio 1954* (Montevideo: Centro de Cooperación Científica de la UNESCO para América Latina), 9-30, on 15-16.
[6] In this respect, especially in terms of research and collaboration, mathematics blossomed comparatively late among the "international" sciences, despite its oft-presumed universality and relative independence of many of the material constraints of other sciences. See DPM, 9-14. This work also develops the historiographical distinction between "international" and "intercontinental" science.

theoretical research in more and further-flung places, forging and refashioning connections between the discipline's global centers and peripheries.[7] Their discipline's transformed scales depended on newly available means of travel and communication and the new institutional connections they allowed, but also on new ways of studying and understanding mathematics that reached to the very words and principles with which mathematicians articulated their theories. Amidst changing geographies and infrastructures, mathematicians debated the nature of mathematics itself—whether it was unified and structured, adaptable and applied, absolute or arbitrary, organic or artificial.[8] What it meant to do mathematics and the means of doing it transformed in tandem, with manifold consequences for mathematicians' ideas and discipline alike.

This essay asks how mathematicians' means shaped their meanings, and how meanings shaped means. I explain the continent-crossing first decade of the theory of distributions by situating the theory's great variety of meanings and interpretations in mathematicians' shifting rhetorical, institutional, material, and other contexts—that is, in mathematicians' variegated means and conditions for developing and sharing theories. Indeed, I argue that the multiplicity of distributions' meanings was essential for the theory's rapid spread across these contexts. Such multiplicity manifested both in the theory's broad conceptual adaptability and in its rhetorical pliability, with disparate interpretations finding common voice in fruitfully ambiguous formulations. Changing means of scholarly exchange and changing approaches to mathematical argument and understanding allowed scholars spanning vast distances to adopt and reconcile a wide range of interpretations of the theory of distributions: conceptual travel facilitated geographic travel and vice versa. Distributions' scholars cast the theory as a banal trick for applied calculations, a difficult intervention in the recent theory of topological vector spaces, a profound realignment of established methods, a radical departure from familiar concepts, and many things in between. They found common ground through metaphors and wordplay, textbooks and abstracts, rhetoric and representation.

The next section places this central finding about heterogeneity and reconciliation

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[7] See DPM; Amy Dahan Dalmedico, "L'essor des Mathématiques Appliquées aux États-Unis: L'impact de la seconde guerre mondiale," *Revue d'histoire des mathématiques* 2 (1996): 149-213; Karen H. Parshall, "Marshall Stone and the Internationalization of the American Mathematical Research Community," *Bulletin of the American Mathematical Society* 46, no. 3 (2009): 459-482; Karen Hunger Parshall, "'A New Era in the Development of Our Science': The American Mathematical Research Community, 1920-1950," in David E. Rowe and Wann-Sheng Horng, eds., *A Delicate Balance: Global Perspectives on Innovation and Tradition in the History of Mathematics, a Festschrift in Honor of Joseph W. Dauben* (Basel: Birkhäuser, 2015), 275-308. On formal efforts toward international mathematical organization, see Olli Lehto, *Mathematics Without Borders: A History of the International Mathematical Union* (New York: Springer, 1998). On the earlier history of internationalization in mathematics, focused principally on Europe, see Karen H. Parshall and Adrian C. Rice, eds., *Mathematics Unbound: The Evolution of an International Mathematical Research Community, 1800-1945* (Providence: American Mathematical Society, 2002).

[8] Leo Corry, *Modern Algebra and the Rise of Mathematical Structures*, 2nd edn, (Basel: Birkhäuser, 2004 [1996]); Amy Dahan Dalmedico, "An image conflict in mathematics after 1945," in U. Bottazzini and A. Dahan Dalmedico, eds., *Changing images in mathematics: from the French Revolution to the new millennium* (London: Routledge, 2001), 223-253; David Aubin, "The Withering Immortality of Nicolas Bourbaki: A Cultural Connector at the Confluence of Mathematics, Structuralism, and the Oulipo in France," *Science in Context* 10, no. 2 (1997): 297-342; Christopher J. Phillips, "In Accordance with a 'More Majestic Order': The New Math and the Nature of Mathematics at Midcentury," *Isis* 105, no. 3 (2014): 540-563; Alma Steingart, *Conditional Inequalities: American Pure and Applied Mathematics, 1940-1975* (Ph.D. dissertation, Massachusetts Institute of Technology, 2013).

within long-running conversations among historians and sociologists of science regarding expertise, communication, and theoretical knowledge. I then set the scene for Schwartz's intervention in the history of functions and the calculus with a brief introduction to the so-called Heaviside and Dirac functions, which were central in several early accounts of Schwartz's theory. After characterizing Schwartz's first presentation of the theory of distributions, I explain the context and implications of his use of wordplay and attention to language, with particular reference to the reframing of mathematics associated with Bourbaki. The mathematical technique of integration by parts, here, exemplifies Schwartz's approach while furnishing an apt description of the kind of social and intellectual consolidation that this approach allowed. The remaining sections detail how the theory of distributions spread to new geographic and intellectual milieux in terms of the foregoing considerations, in relation to new institutional, geopolitical, and other conditions. I conclude by revisiting the problems of meaning and ambiguity for histories of theoretical knowledge, arguing for the central place of historical actors' ongoing labor of partial integration.

[FIRST LEVEL HEADING] EXPERTISE, EXCHANGE, AND THE PLURALITY OF THEORY

A theory can be many things to many people, but to each of a theory's users and exponents it tends to be, in itself, just one thing. Different theorists with different emphases, framings, priorities, and commitments can agree or disagree, adopt or adapt, understand or misunderstand, all while believing themselves be engaging one and the same theory. This tension between a unitary conception of scientific knowledge and its plural manifestation has been fundamental to the history and sociology of science. Viewed in social context, theories are not just collections of definitive claims, premises, methods, and implications, but also social and institutional relationships, ways of simultaneously organizing both knowledge and its producers. Often, theories and associated technical systems join together actors across a spectrum of social roles and positions, giving rise to identifiable patterns relating social structure to theoretical understanding.[9] Even actors in comparable social circumstances must reconcile divergent piecemeal understandings, and I have elsewhere argued that this kind of medium-dependent labor is fundamental to both the social and intellectual practice of present-day mathematical research.[10]

Informed by this work, I take the perspective here that theories are best understood as social systems built around relationships of partial mutual understanding, and theoretical labor is best interpreted in terms of generating and reconciling senses of mutual understanding that underwrite those relationships. I mean this in a strong sense: where used as an analytic term, one may (in principle) read "theory of distributions" as "social system built around relationships of partial mutual understanding of distributions." By placing social relationships, rather than conceptual objects, at the center of the analysis, this view of

[9] E.g. what Donald MacKenzie has called the "certainty trough" of trust in technical systems. See Donald MacKenzie, *Inventing Accuracy: A Historical Sociology of Nuclear Missile Guidance* (Cambridge, MA: MIT Press, 1990), 370-372. MacKenzie discusses the certainty trough for mathematical proof, specifically, in *Mechanizing Proof: Computing, Risk, and Trust* (Cambridge, MA: MIT Press, 2001).

[10] Michael J. Barany and Donald MacKenzie, "Chalk: Materials and Concepts in Mathematics Research," in Catelijne Coopmans, Michael Lynch, Janet Vertesi, and Steve Woolgar, eds., *Representation in Scientific Practice Revisited* (Cambridge, MA: MIT Press, 2014), 107-129. See also Alma Steingart, "A group theory of group theory," *Social Studies of Science* 42, no. 2 (2012): 185-213.

mathematical theories emphasizes a relativist epistemology of mathematics.[11] Since the only empirical basis for establishing mutual comprehension is the social activity it licenses, my historical interpretation of the past's mathematical understandings shall stress how claims about ideas and methods constituted or transformed communities of interaction.

Knowledge of distributions was highly differentiated, ranging from recognition to familiarity to connoisseurship to interactional competence to pedagogical competence to research expertise or virtuosity. Though many found the theory of distributions easier to learn than other theories of comparable reach and novelty, to use distributions in one's research could still require weeks or months of special study on top of years of relevant specialized training. But to appreciate distributions—to "get a sense" of the theory—required little more than experience with undergraduate calculus, an experience professional mathematicians could readily take for granted. The theory's adoption depended on communities of mathematicians, physicists, and engineers with substantially different kinds of understanding being able to interact fruitfully in a range of settings and for a variety of purposes. If the theory of distributions appeared as just one thing, it owed to the vast and circumstance-dependent labor required to see unity in the great diversity of texts and contexts, of knowledges and practices, that came to be associated under its banner.

Historians and sociologists of science may recognize this kind of claim under such familiar rubrics as "trading zones" or "boundary objects."[12] Whenever distinct communities converge around common projects, they translate their different interests and understandings by way of entities that can tie the communities together while sustaining different meanings for each. As with diagrams or formulae in physics or finance, the formalisms scholars used to represent distributions succeeded as bridges between communities because their respective users could understand them very differently, while continuing to hold them in common.[13] Moreover, just as some kinds of data, specimens, or apparatus in the natural sciences travel better than others, so too do some loci of representation in theoretical disciplines like mathematics.[14] Here, the established historiography on the transformation of scientific knowledge through textual transmission, translation, and interpretation helps to explain both the pluralization and consolidation of distributions as a theory rooted in polysemic renderings of mathematical phenomena.[15]

Distributions' interpretive pliability has haunted both historical discussions of the theory itself and the theory's limited historiography. The history reported here turns on

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[11] See Michael J. Barany, "Of Polyhedra and Pyjamas: Platonism and induction in meaning-finitist mathematics," in Elizabeth de Freitas, Nathalie Sinclair, and Alf Coles, eds., *What is a mathematical concept?* (Cambridge: Cambridge University Press, 2017), 19-35.

[12] Peter Galison, *Image and Logic: A Material Culture of Microphysics* (Chicago: University of Chicago Press, 1997), ch. 9; Susan Leigh Star and James R. Griesemer, "Institutional Ecology, 'Translations' and Boundary Objects: Amateurs and Professionals in Berkeley's Museum of Vertebrate Zoology, 1907-39," *Social Studies of Science* 19 (1989): 387-420.

[13] E.g. Galison, *Image and Logic* (ref. 12); David Kaiser, *Drawing theories apart: the dispersion of Feynman diagrams in postwar physics* (Chicago: University of Chicago Press, 2005); Donald Mackenzie, "An Equation and Its Worlds: Bricolage, Exemplars, Disunity and Performativity in Financial Economics," *Social Studies of Science* 33, no. 6 (2003): 831-868; Paul Erickson, *The World the Game Theorists Made* (Chicago: University of Chicago Press, 2015).

[14] Cf. Peter Howlett and Mary S. Morgan, eds., *How Well Do Facts Travel? The Dissemination of Reliable Knowledge* (Cambridge: Cambridge University Press, 2010).

[15] E.g. Catherine Goldstein, *Un théorème de Fermat et ses lecteurs* (Saint-Denis: Presses universitaires de Vincennes, 1995); Caroline Ehrhardt, *Itinéraire d'un texte mathématique: Réélaborations d'un mémoire de Galois au XIXe siècle* (Paris: Hermann, 2012); Marwa S. Elshakry, "Knowledge in Motion: The Cultural Politics of Modern Science Translations in Arabic," *Isis* 99, no. 4 (2008): 701-730; James Secord, "Knowledge in Transit," *Isis* 95, no. 4 (2004): 654-672.

several examples of methods and concepts being identified as distributions in disguise, a stipulation aided by specific linguistic practices discussed below. The first extended study of the history of distributions focused on the massive tangle of conceptual and genealogical relationships among the many ideas and practices that came to be associated with distributions in the theory's early years.[16] For Schwartz's critics, the range of apparently equivalent approaches exogenous to his own was evidence that the theory was merely derivative—consisting of old ideas in new clothing and reflecting differing national or personal styles of mathematics.[17] Schwartz's defenders (including Schwartz himself), conversely, painted the theory's unification of so many exogenous approaches as evidence of its profundity, while bootstrapping a constituency for the theory of distributions by laying claim to those already versed in the other approaches.[18] Recent scholarship has followed the theory between different scholarly communities to draw conclusions about those communities' philosophical values or forms of collective engagement.[19]

At the same time, both historical actors and historians have tended to treat the theory of distributions as *just one thing*, albeit one found in many guises. Seeing unity, like recognizing difference, must be understood as the result of situated interpretation. The same precept applies to what a theory is and what it means to know a theory. The history of distributions hinges on what it means to know *the same thing* differently across times and places.

[FIRST LEVEL HEADING] FUNCTIONAL DISTINCTIONS

One way—likely the most common one historically—to understand the theory of distributions is as a generalization of two notions fundamental to calculus and mathematical analysis: functions and their derivatives. In this respect, Schwartz joined a tradition dating to the first century of Newton's and Leibniz's calculus.[20] By the mid-eighteenth century, Leonhard Euler established a widely followed interpretation of functions as relations between independent and dependent variable quantities, with derivatives as the rate of change of a

[16] LPD. See also John Synowiec, "Distributions: The Evolution of a Mathematical Theory," *HM* 10 (1983): 149-183.
[17] E.g. Salomon Bochner, Review of Laurent Schwartz, Théorie des distributions, vols. 1 and 2. *Bulletin of the American Mathematical Society* 58, no. 1 (1952): 78-85; Jean-Michel Kantor, "Mathematics East and West, Theory and Practice: The Example of Distributions," *MI:* 39-46; S. Kutateladze, "Some Comments on Sobolev and Schwartz," *MI:* 51.
[18] E.g. Jean Dieudonné, Review of LPD, *AMM* 91, no. 6 (1984): 374-379; Colette Anné, Jean-Pierre Bourguignon, Claude Viterbo, eds. *Supplément au numéro 98 de la Gazette des mathématiciens*, 2003; MGC, ch. 6; Peter Lax, "The Reception of the Theory of Distributions," *MI:* 52.
[19] Klaus-Heinrich Peters, "Mathematische und phänomenoligische Strenge: Distributionen in der Quantenmechanik und -feldtheorie," in Karl-Heinz Scholte and Martina Schneider, eds., *Mathematics meets physics: A contribution to their interaction in the 19th and the first half of the 20th century* (Frankfurt am Main: Verlag Harri Deutsch, 2011), 373-393; Peters, *Der Zusammenhang von Mathematik und Physik am Beispiel der Geschichte der Distributionen: Eine historische Untersuchung über die Grundlagen der Physik im Grenzbereich zu Mathematik, Philosophie und Kunst* (Ph.D. dissertation, Hamburg University, 2004). PLS; Anne-Sandrine Paumier, "Le séminaire de mathématiques: un lieu d'échanges défini par ses acteurs. Incursion dans la vie collective des mathématiques autour de Laurent Schwartz (1915-2002)," *Philosophia Scientiae* 19, no. 2 (2015): 171-193.
[20] LPD. See also J. R. Ravetz, "Vibrating Strings and Arbitrary Functions," in *The Logic of Personal Knowledge,* (London: Routledge, 1961), 71-88; A. P. Youschkevitch, "The Concept of Function up to the Middle of the 19th Century," *AHES* 16, no. 1 (1976): 37-85; Garrett Birkhoff and Erwin Kreyszig, "The Establishment of Functional Analysis," *HM* 11 (1984): 258-321.

dependent variable with respect to a specified independent variable.[21] Most subsequent interpretations of functions and derivatives followed suit, reinterpreting variables, rates of change, and other terms and concepts accordingly.

History's various interpreters of functions and derivatives dwelled, in particular, on the relative foundational statuses of different forms of representation, ranging from graphs and diagrams to explicit or implicit formulae to different kinds of symbolic expressions of abstract relations.[22] As mathematicians offered new definitions of the central terms of the calculus, this history of adaptation and reinterpretation was itself an important feature of their rhetorical and conceptual interventions: by the nineteenth century, mathematicians recognized the calculus as a subject whose changing fundamental notions manifested the changing state of mathematics itself. Sometimes this history was a measure of mathematicians' progress.[23] Other times, as for Émile Picard at the opening of the 1920 International Congress of Mathematicians, "analyzing the subtlest recesses of the idea of a function" represented the cloistered predilections of "incorrigible idealists," whose attitude Picard found untenable after the Great War.[24]

For most theorists of differential equations since Euler, functions and derivatives presented a persistent dilemma. A perfectly valid (even useful) function might have discontinuities or irregularities, corresponding to rates of change that could be infinite or even indefinable, which created havoc however one defined derivatives. Worse, many such non-differentiable functions were mathematically significant, and indeed often practically useful for specialized calculations. So one had, in effect, to choose between prioritizing the functions of interest or one's ability to take derivatives unproblematically. If favoring the potentially non-differentiable functions, one could attempt to work around its irregularities in a piecemeal fashion. If seamless differentiation was more important, one could focus exclusively on smooth functions that lacked troublesome irregularities altogether but somehow approximated the non-differentiable functions. Either way, something was lost.

Promoting his theory, Schwartz paid special attention to one situation where physicists and engineers had, in their own ways, grappled notably with the tradeoff between irregular but supremely useful functions and their rigorously tractable but less useful cousins: their use of the Heaviside and Dirac functions. I shall present them now in a somewhat more didactic mode than I use for other mathematical discussions in the rest of this article, where I have attempted to hew as closely as possible to actors' often ambiguous and differentially understood presentations. Here, however, a less historically faithful exegesis shall help illustrate the kinds of interactions among meanings, techniques, and stipulations about rigor and validity that surface in other guises in the ensuing historical discussion.

British electrical engineer Oliver Heaviside's eponymous function seems at first glance about as simple as a function could be. It relates an independent variable (say, x) to a dependent variable (say, y) by giving y the value 0 when x is negative and 1 when x is positive. If you interpret x as time, the Heaviside function can be interpreted to represent flipping a switch at time $x=0$, with $y=0$ meaning the switch is off and $y=1$ meaning the switch is on. On a Cartesian graph, the Heaviside function is a flat line at altitude $y=0$ to the

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21 Henk J. M. Bos, 'Differentials, Higher-Order Differentials and the Derivative in the Leibnizian Calculus,' *AHES* (1974) 14, pp. 1-90.

22 E.g. Michael J. Barany, "God, King, and Geometry: Revisiting the Introduction to Cauchy's *Cours d'Analyse*," *HM* 38, no. 3 (2011): 368-388.

23 See Joan L. Richards, "Historical Mathematics in the French Eighteenth Century," *Isis* 97, no. 4 (2006): 700-713.

24 Émile Picard, "Séance d'Ouverture du Congrès: Allocution de M. Émile Picard," in Henri Villat, ed., *Comtes Rendus du Congrès International des Mathématiciens (Strasbourg, 22-30 Septembre 1920)* (Toulouse: Édouard Privat, 1921), xxvi-xxix, on xxviii-xxix.

left of the y-axis and a flat line at altitude $y=1$ to the right, forming a "step" shape.

Heaviside used this function as a basic element for a method of symbolic operations that many found enormously effective for a wide variety of calculations, but which proved controversial for its lack of adherence to prevailing norms of mathematical rigor.[25] To see the mathematical trouble, consider the Heaviside function's derivative. Almost everywhere, the y value of the function does not change at all, and so the function has a derivative of 0 (meaning a null rate of change). But at the crucial point $x=0$, the function jumps infinitely quickly between the values 0 and 1 of the dependent variable y. Because it represents the rate of change necessary to jump precisely from 0 to 1 in this instant, this function is sometimes called the "unit impulse." This one infinite point made calculations with the Heaviside function's derivative unrigorous, critics held.

Matters were worse for the derivative of the derivative: 0 except at $x=0$, and when $x=0$ rising infinitely quickly to reach the derivative's infinite value at that point before falling (with x still 0) infinitely quickly to return to the derivative's subsequent value of 0—that is, the Heaviside function's second derivative when $x=0$ was positive infinity and *simultaneously but subsequently* negative infinity. Corresponding to the first derivative's "unit impulse," this second derivative is sometimes called the "unit dipole" or "unit doublet," since its equal and opposite values—if "values" can be said to apply here—are precisely the acceleration needed to produce the instantaneous jump from 0 to 1 reflected in the unit impulse.

Some decades after Heaviside debuted his symbolic calculus, Paul Dirac introduced his own symbolic calculus using the Heaviside function and its derivatives. Dirac's calculus furnished an important method for atomic physics but was likewise criticized as lacking proper mathematical foundations. He denoted the Heaviside function's derivative with the Greek letter δ, and others subsequently referred to δ as "Dirac's function," "Dirac's delta," "the Dirac function," or similar variations.

If one insisted on avoiding paradoxical infinities, one could imagine the Heaviside function and its derivatives as the limits of corresponding sequences of functions that were perfectly smooth and hence unproblematically differentiable. In place of the Heaviside function itself, picture an 'S'-shaped curve that replaces the instantaneous leap of the Heaviside function at $x=0$ with a smooth take-off when x has merely a very tiny negative value, followed by a rapid (but not infinitely so) rise toward $y=1$ and a smooth landing at $y=1$ as x reaches a tiny positive value. The derivative then looks something like a very tall bell centered around the y-axis, and the second derivative features two tall bells: one peaking high and to the left of the y-axis and the other dipping low to the right of the y-axis. These pictorial approximations (and a corresponding set of customary equations) represented one response to the function-derivative dilemma, letting one apply derivatives just to bounded and differentiable functions at the cost of a great many more manipulations to access the limiting results of the Heaviside and Dirac calculi themselves.

A more common response to the dilemma, however, involved learning a system of rules and conventions for manipulating the Heaviside, Dirac, and related functions directly using ordinary formalisms for functions and their derivatives. In some circumstances, these functions could be manipulated just like any ordinary non-paradoxical function. In other circumstances, a few principles combined with familiar rules of calculation to dictate how the function could be used. For instance, to compute the integral of the Dirac function one used

[25] Bruce J. Hunt, "Rigorous Discipline: Oliver Heaviside Versus the Mathematicians," in Peter Dear, ed., *The Literary Structure of Scientific Argument: Historical Studies* (Philadelphia: University of Pennsylvania Press, 1991), 72-95; Jesper Lützen, "Heaviside's Operational Calculus and the Attempts to Rigorise It," *AHES* 21, no. 2 (1979): 161-200.

the values of its antiderivative (the Heaviside function) and the formula from the Fundamental Theorem of Calculus, ignoring that this theorem did not rigorously apply to the Dirac function.[26] In yet other circumstances, users had to learn to rule certain questions or manipulations out of bounds, as treading impermissibly on the functions' paradoxical qualities.

That is, some ways users were accustomed to working with ordinary functions translated easily to these other functions. Some ways translated somewhat, under the appropriate cautions and with some necessary contrivances. Some other ways—such as examining them graphically or computing with point-values—were inapplicable or applied only in rough approximations. Users drew techniques and understandings from their training and aims, recognizing paradox-laden functions as cognate to unproblematic ones in ways specific to the contexts and purposes of their use. Even when using similar symbolic representations for such functions, meanings attributed to them and justifications for their uses could vary. Where an electrical engineer might see an equation as representing an idealized electric potential, a quantum physicist may rationalize a similar equation as derived from wave functions. Similar manipulations could come from distinct rules and trainings. Paradoxes and exceptions, too, were susceptible to different conceptualizations. Even such simple examples hid worlds of divergent meaning.

In many respects, this kind of conceptual and practical accommodation is reminiscent to developments in the history of numbers that have been studied historically at greater length.[27] The advent of negative numbers, imaginary and complex numbers, and infinitesimals each owed in part to tentative accommodations between mathematical principles and practical ends. Each raised problems about rigor, generality, and foundational validity that could be grounds for philosophical dispute. Each was defended in some quarters as a useful (if not necessarily grounded) fiction, justified through its utility in calculation. Eventually, new proposals about rigor and foundations proved their worth in their capacity to integrate such once-exotic or exceptional numerical entities as part of a coherent system. So, too, did Schwartz derive his theory's merit (and lay claim to its potential constituencies) from its systematic incorporation of Heaviside and Dirac calculi that had been variously accommodated in other terms. While many aspects of the story of distributions are specific to their historical times and places, they were not without models and patterns from the past that shaped distributions' understandings both directly and indirectly. This history rhymes, and its protagonists could hear it.

[FIRST LEVEL HEADING] A THEORY IS BORN

Schwartz first sketched his theory for public consumption in an eighteen-page article in the 1945 annual volume (printed in 1946) of the *Annales de l'université de Grenoble*, the in-house publication series of the university where Schwartz held his first academic

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[26] The result is $\int_a^b \delta(x)\, dx = H(b) - H(a)$, which equals 1 if the interval [a,b] includes 0 (that is, covers the 'step' in the step function) and equals 0 otherwise. I have simplified the foundational issues here. There are formulations of integration that give a rigorous definition for the integral of δ, but δ is not properly integrable using either Riemann or Lebesgue integration, and conventional means of accounting for δ prior to the theory of distributions did not always account as well for its derivatives.

[27] Esp. Gert Schubring, *Conflicts between Generalization, Rigor, and Intuition: Number Concepts Underlying the Development of Analysis in 17-19th Century France and Germany* (New York: Springer, 2005). I thank Olivier Darrigol for prompting an explicit discussion of this historical parallel.

appointment.[28] The article itself, which he later called his "propaganda tract,"[29] seems to have circulated mainly in the offprints Schwartz distributed to an ever growing circle of mathematicians. More significantly, it is the earliest available record of the template for many of Schwartz's initial expositions of the theory of distributions, including his debut lectures on the theory in his January-April 1946 Cours Peccot, a prestigious half-century-old Paris lecture series for young mathematicians organized by the Collège de France, and his July 1946 Bourbaki presentation.[30]

The article's page long overture signaled Schwartz's ambitions by beginning with physicists and their calculations, not mathematicians and their proofs. Or rather, with a caricature of physicists as freewheeling artisans of the Dirac calculus, loyal to fruitful calculation but not to mathematical rigor. Though the latter part of his article's title promised "mathematical and physical applications," Schwartz put all of the claimed physical applications in this brief introduction, which centered on the alleged absurdities of the Dirac function and its derivatives. "Since the introduction of the symbolic calculus," Schwartz began, "physicists have commonly made use of certain notions or certain formulas for which the success is indisputable, while not being mathematically justified."

Schwartz described the Heaviside function (without the eponym) and then introduced (under Dirac's name) the key properties of the Dirac function: "null for $x \neq 0$, equal to $+\infty$ for $x=0$," and with a definite integral of 1 over its whole domain. "Such an 'abuse of language,'" wrote Schwartz, "is despite everything incompatible with the habitual notion of function and derivation!" He announced that his article would summarize a forthcoming monograph (an albatross of a textbook that would take him the next half decade to complete) that "will provide a complete justification to the preceding language," and so "rehabilitate" the Dirac formalism for mathematicians and physicists alike.

One should hesitate to take at face value Schwartz's assertion about the Dirac function's faults. To be sure, Schwartz was not alone among mathematicians in declaring the need for a completely rigorous justification of the Dirac function. All the same, physicists, engineers, and even mathematicians had developed a wide range of informal and formal justifications for their Dirac calculi. With approximations, gestures, illustrations, and especially with rules and conventions of symbolic manipulation, these scholars used the Dirac function without being apparently beleaguered by supposed foundational difficulties. The Dirac function's purported lack of mathematical justification *despite* its usefulness was not a pre-given rationale or state of affairs, but rather something Schwartz had to assert, in part by ruling its many manifest justifications to be non-mathematical or insufficiently rigorous.

The Dirac function's derivatives supplied the template for Schwartz's proposed rehabilitation, both narratively and conceptually. While the Dirac function was not itself a conventional function, its users could understand it as the derivative of the conventional Heaviside function, which was not differentiable at the critical point $x=0$. To justify an unconventional derivative of a conventional function, Schwartz interpreted the Dirac function as a member of a broader class of objects called measures. Schwartz then repeated this justification through generalization by portraying the Dirac function's derivative, which was not a conventional measure, as one of a still broader kind of entity called a linear functional.

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[28] Laurent Schwartz, "Généralisation de la notion de fonction, de dérivation, de transformation de Fourier et applications mathématiques et physiques," *Annales de l'université de Grenoble* 21 (1945): 57-74. On Schwartz's wartime and early postwar situation, see MGC, 165-167, 190-191, 202, 211; BPL, §2.

[29] MGC, 239.

[30] On the Cours Peccot, see MGC, 240; BPL, §2. The latter ties Schwartz's Cours Peccot selection directly to his wartime Bourbaki connections.

He then defined distributions as a certain kind of linear functional (denoted *T*), one that assigned a number to each infinitely differentiable function (denoted φ) whose value was 0 outside of a bounded region. These criteria for φ obviated troublesome considerations about differentiability and boundary terms. When the distribution was a measure (denoted μ) or a function (denoted *f*), the number was determined by the definite integral of φ weighted by the function or measure—in Schwartz's symbolic notation $f(\varphi) = \int f(x)\varphi(x)dx$ or $\mu(\varphi) = \int \varphi(x)d\mu$. Under the right conditions distributions could be measures and measures in turn could be functions, but each type of object included examples that were not of the less general type.

As contemporaries recognized, this sequence of generalizations rhymed with the just-noted pattern in the history of number concepts.[31] Corresponding to functions and derivatives, earlier scholars grappled with numbers and the operations of arithmetic and algebra. Counting numbers (0, 1, 2, 3,…) were unproblematic under addition, but subtraction required generalizing to all integers, including once-controversial negative numbers. As with the generalization from functions to measures, when addition problems involved only counting numbers, everything worked as before, but arithmetic with negative numbers expanded what could be treated systematically and coherently. Integers accommodated addition, subtraction, and multiplication, but division called for fractions, or rational numbers. Finally, mathematicians allowed a complete and systematic framework including the extraction of square roots by admitting imaginary and complex numbers. Here, the chain of generalizations came to a close: complex numbers did the job not just for square roots but for *any* algebraic operations. Mathematicians learned to see this as evidence that the complex numbers were a natural culmination of numerical generalizations, but this did not stop efforts to find and advocate for yet further fruitful generalizations, such as quaternions or infinitesimally-enriched continua.

Starting with the Heaviside function, successive derivatives required successive generalizations to accommodate them. Then, as with the algebraic culmination in complex numbers, the generalizations miraculously stopped at linear functionals: every distribution had a derivative that was also a distribution, and was itself the derivative of a distribution. Just as no operation of algebra would ever need more than the complex numbers, no operation of the calculus (at least as construed here) would require more than distributions. But what exactly did differentiation mean when a distribution was not also a function? Here, Schwartz followed his justification-through-generalization in reverse. Any operation applied to a distribution that happened to correspond to a function had to give the same result as the corresponding conventional operation from integral and differential calculus. Thus, the distributional derivative of a differentiable function was to match its conventional derivative. Schwartz then defined operations for arbitrary distributions (which did not, in general, have conventional derivatives) by translating a usual calculus presentation into one intelligible for linear functionals and then declaring that to be the general definition for distributions.

This reasoning rested on a combination of formal and informal wordplay. Arbitrary distributions were like functions, Schwartz implied, so operations with the one should look and feel like those with the other, using similar formalisms and terminology while calling upon similar intuitions. To define differentiation, Schwartz began with a conventionally differentiable function *f* in *n* variables and its derivative in the direction of the *i*th independent variable x_i (denoted with an apostrophe and a subscript variable). As a linear functional, this gave the expression:

$$f'_{x_i}(\varphi) = \iint \dots \int f'_{x_i}(x_1, \dots, x_n)\varphi(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

[31] Bohr, "Address" (ref. 4), 132.

Then, Schwartz asserted, “there is no difficulty to integrate by parts,” applying a routine technique from the conventional calculus to rewrite the expression as:

$$f'_{x_i}(\varphi) = -\iint \dots \int f(x_1, \dots, x_n)\varphi'_{x_i}(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

or, in the idiom of linear functionals, $f'_{x_i}(\varphi) = f(-\varphi'_{x_i})$.[32] The subtle but important difference between the two integral expressions involved removing the derivative from the function *f* and, along with the introduction of a minus sign, applying the derivative instead to the function φ. Rendered in the compact formalism of linear functionals, stripped of the visual trappings and technical provisos of integration, the transformation was striking. Schwartz used this formal sleight-of-hand to define the derivative of a distribution as $T'_{x_i}(\varphi) = T(-\varphi'_{x_i})$. The left side of this equation had no prior definition—the whole point of the endeavor was that *T* was not differentiable—but the right side appeared mathematically sound because the functions denoted φ were presumed differentiable, so there was no new difficulty for *T* to act on their derivatives.

Integration by parts began, on the fifth page of Schwartz’s article, as a familiar computational technique for evaluating integrals of conventionally differentiable functions. By characterizing that technique instead as an operation in the abstract calculus of functionals, within half a page he could declare that integration by parts “permits the generalization of the notion of a derivative” and so he could correspondingly apply the formal operation of integration by parts to distributions for which a conventional calculus interpretation was impossible.[33] To accomplish this, Schwartz elided the significant conceptual distinctions between the superficially similar equations for functions and distributions, justifying the latter equation by portraying it as the same as the former. He defined distributional derivatives by turning integration by parts from a technique into a metaphor. This only worked by stripping both expressions of their mathematical justifications and conditions of validity and interpreting them instead through their formal appearances—a symbolic pun, wordplay.

Such a pivotal use of what amounted to wordplay was plain to Schwartz’s contemporaries. After Schwartz shared his theory late in 1947 with Marcel Riesz in Lund, Sweden, the latter (in Schwartz’s telling) counseled him “to watch out for what is nothing but … wordplay [jeu de mots],” as the “very beautiful theory” still “wants for the effort of becoming deepened over the years.”[34] Riesz recognized wordplay as both a resource and a danger, offering a beautiful formulation of considerable potential, provided one committed to the extended labor of realizing the theoretical depths that wordplay itself could only suggest. As the ensuing account shows, Schwartz both heeded and skirted this advice, relying at times on his wordplay-mediated ability to promote the theory superficially and even to argue that its essence was reflected in these superficial expositions. Indeed, Schwartz debuted his theory at a time when wordplay seemed itself more and more central to the fundaments of mathematical investigation.

[FIRST LEVEL HEADING] MAKING SENSES OF DISTRIBUTIONS

Schwartz’s formal prestidigitation represented an approach to mathematical argument increasingly visible in the twentieth century, one evident in a dual meaning of the word “sense” (“sens” in French) in Schwartz’s 1945 article. The first meaning came in references

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[32] Note that here the integral is implicitly over the entire domain of the function, with the vanishing conditions on φ allowing Schwartz to omit boundary terms without comment.

[33] Schwartz, “Généralisation,” (ref. 28), 61.

[34] Schwartz, “Premiers travaux” (ref. 3), 118. A translation of Schwartz’s fuller remarks in context is in BPL, §6.

to “the usual sense” of terms, indicating a way of understanding and using a term that might elsewhere be understood and used differently. Beginning in the late nineteenth century, a significant minority of mathematicians and philosophers had devoted special attention to the relationships between mathematics and the linguistic means and metamathematical frameworks by and through which it was articulated.[35] Though most mathematicians did not share the full measure of that minority’s linguistic and philosophical preoccupations, they nevertheless increasingly specified precise interpretive contexts (or “senses”) for mathematical terms as a routine feature of their expositions. Such specifications allowed mathematicians to recycle terms deliberately, taking rhetorical advantage of the terms’ other meanings while stressing the technical differences of their new proposed uses.

By renaming the context of derivatives while keeping the same terminology for the operation of differentiation, Schwartz signaled that the essence of differentiation would be unchanged. Conversely, by replacing “functions” with the new term “distributions,” he suggested a more substantive conceptual break—one not wholly embraced by the many who later translated distributions as “generalized functions.” The term “distributions” was itself borrowed from probability, physics, and measure theory, and Schwartz’s distributions shared many conceptual and practical features with the corresponding distributions in these other settings. To emphasize Schwartz’s particular context for “distributions,” especially in situations where other formulations of distributions might apply, mathematicians came to refer to “Schwartz distributions” or “distributions in the sense of Laurent Schwartz.”

Schwartz’s second usage of “sense” in the 1945 paper came from claims that statements “have a sense” or “make sense” (“a/ont un sens”), or conversely that they did not make sense. A formulation had a sense if it was completely justifiable, if not necessarily completely justified in the work at hand. For example, Schwartz asserted that “here, [derivatives] dA/dy and dB/dx always have a sense qua distributions.” This emphatic usage of sense was more particular to those writing about mathematical foundations, for whom it stressed the presence or absence of a coherent basis for an expression that might otherwise be assumed valid simply by virtue of its use in a mathematical argument.

The ambition to guarantee that every mathematical notion had a sense was a key pillar of the Bourbaki collaboration’s central project, a series of textbooks titled *Elements of Mathematic* (singular).[36] The introduction to the first volume stipulated that the group’s texts were to be “formalized,” meaning that they could in principle be expressed in a simple language with “a syntax consisting of a small number of unbreakable rules” that guaranteed the sensibility of every term. However, for the Bourbaki collaborators as for other mathematicians, “having a sense” was almost invariably a promissory claim rather than a meticulously demonstrated conclusion. Instead, Bourbaki explained, “In general [the mathematician] is content to bring the exposition to a point where his experience and

[35] Corry, *Modern Algebra* (ref. 8); Ivor Grattan-Guinness, *The Search for Mathematical Roots, 1870-1940: Logics, Set Theories and the Foundations of Mathematics from Cantor through Russell to Gödel* (Princeton: Princeton University Press, 2000); Herbert Mehrtens, *Moderne – Sprache – Mathematik: eine Geschichte des Streits um die Grundlagen der Disziplin und des Subjekts formaler Systeme* (Frankfurt am Main: Suhrkamp, 1990). Philosophies from this period continue to animate investigations in the philosophy of science. Of particular note with respect to my present argument is Emily R. Grosholz, *Representation and Productive Ambiguity in Mathematics and the Sciences* (Oxford: Oxford University Press, 2007), 16-24, 33-60, 126-139, 227-284, which takes the positive philosophy of Rudolf Carnap (e.g. *Der logische Aufbau der Welt,* Berlin: Weltkreis, 1928) as a starting point.

[36] See Leo Corry, “Writing the ultimate mathematical textbook: Nicolas Bourbaki’s *Élements de mathématique*,” in Eleanor Robson and Jacqueline Stedall, eds., *The Oxford Handbook of the History of Mathematics* (Oxford: Oxford University Press, 2009), 565-588.

mathematical flair tell him that translation into formal language would be no more than an exercise of patience (though doubtless a very tedious one)."[37] In place of tedious formalities, Bourbaki permitted "abuses of language" in order to make texts legible by assuming terms and claims to be sensible without rigorously justifying them.[38]

Bourbaki did not, of course, have a monopoly on either "sensibility" or mathematical foundations. After Bourbaki collaborator Jean Dieudonné laid out the Bourbaki philosophy of axiomatic mathematical method at a 1949 *Colloquium of Mathematical Philosophy* at the Sorbonne, in Paris, the colloquium secretary and Sorbonne professor Arnaud Denjoy replied with a lengthy denunciation in the colloquium proceedings.[39] Dieudonné, for his part, asserted that "The axiomatic method is tightly linked to the nature of deductive reasoning itself" and recognized a historical ebb and flow of axiomatic reasoning with "periods of expansion" giving rise to controversy-filled "periods of clearing [défrichement] … to restore the edifice to order" (47). Creativity and rigor lay in tension, but at the same time "the axiomatic is perfectly compatible with a certain sort of intuition" and mathematicians' insistence on "sensible experience" "rested on a psychological illusion" about the underlying source of their convictions (50). For Dieudonné, axiomatic methods represented a means to support and clarify creative mathematics, not a stricture to rule out play and inspiration.

Denjoy, by contrast, saw in Dieudonné's regard for axioms and logic a "fierce repudiation of every regard for testimony of sensible origin" (8). Noting that "A French mathematical school already enjoys universal notoriety, assembling itself under the emblems and pennants of a mythical chief, Nicolas Bourbaki," Denjoy rued that "Problems of definition and existence, barren branches of a half-dead tree, flower again when one evokes Nicolas Bourbaki" (6). The two core preoccupations reflected in Schwartz's double meaning of sense—definition (the sense of a term) and existence (having a sense)—must for Denjoy be subordinated to the only sense that mattered, "the source of original intuitions": one's sense of "a world composed of experiences borrowed from the concrete" (*ibid.*). Denjoy elaborated this into a specific claim about the source of sense and significance in mathematical symbols: "Symbols, their associations ordered after set rules, do not come into possession of their signified sense except in relation to a mind capable of grasping them. … the abstract must always be clarified by the concrete" (15-16).

Other critics saw in Bourbaki's approach to language little more than a coy evasion. Paul Halmos, introducing the collaboration to readers of *Scientific American,* averred that Bourbaki's insistence on calling out "abuses of language" showed Bourbaki's "slightly contemptuous" view of conventional terminology while preserving the collaborators' ability to use those same familiar terms.[40] But for Bourbaki this was precisely the point: mathematicians should not eschew conventional terms but must always pay attention to the sense (in Schwartz's first usage) in which they made sense (in Schwartz's second usage). Bourbaki's collaborators reveled in puns and parodies, and their intense, jocular meetings

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[37] Nicolas Bourbaki, *Elements of Mathematic: Theory of Sets* (Paris: Hermann, 1968), 7-8; [N. Bourbaki, *Éléments de Mathématique, Livre 1: Théorie des Ensembles* (Paris: Hermann, 1954), 1-2].

[38] Bourbaki, *Elements* (ref. 37), vi, 11 [iii, 6].

[39] Jean Dieudonné "L'Axiomatique dans les mathématiques modernes," in Raymond Bayer, ed., *XVII Congrès International de Philosophie des Sciences, Paris, 1949,* vol. 3 (Paris: Hermann, 1951), 47-53; Arnaud Denjoy "Rapport général sur les travaux du colloque de philosophie mathématique," *idem,* 3-22. I encountered this reference in a 21 Feb 1952, letter from Buenos Aires physicist and philosopher Mario Bunge to the Massachusetts Institute of Technology's Dirk Struik, generally approving of Denjoy's "unusually violent attack…against the formalism of the Bourbaki group." Papers of Dirk Jan Struik, MC.0418, Institute Archives and Special Collections, Massachusetts Institute of Technology, Cambridge, Massachusetts, Box 7, folder "A-C."

[40] Paul R. Halmos, "Nicolas Bourbaki," *Scientific American* 196 (1957): 88-99, on 93-94.

demanded virtuosic displays of interpretive versatility across fields of mathematics and their corresponding nuances of language.[41] Their rejection of naïve or unconsidered usages was as much a social and cultural norm as an intellectual and philosophical one—a social and cultural norm, it should be said, that could be deliberately exclusive and mocking, even cruel, to outsiders.

When Schwartz, borrowing Bourbaki's turn of phrase, referred to the Dirac function in his 1945 introduction as an "abuse of language," he was not condemning it but rather calling attention to its need for a sense—a sense given to it at long last by distributions. Giving something such a sense could itself be a sort of abuse of language, an invocation promising a justification whose validity rested more on habit and experience than a tedious exercise of patience. A full explication of what a term or expression meant as a statement about distributions could be onerous, and was rarely necessary to make a credible claim to sensibility. With practice, distributions' users found it easy to manipulate terms subject to the premises and implications of Schwartz's theory by, for instance, deploying integration by parts metaphorically, and they could confidently recognize when and how it was appropriate to do so. To those not versed in the theory's substantial theoretical underpinnings, results sometimes seemed to come more from wordplay than rigorous mastery.

[FIRST LEVEL HEADING] SENSE AND NONSENSE

The changing place of sense in modern mathematics found voice not just in mathematical theory but in mathematical farce. In 1937, Ralph P. Boas and some Princeton companions started a dinner-table game of devising mathematical methods for catching lions.[42] Their 1938 article in the *American Mathematical Monthly,* published under a pseudonym (H. Pétard) in homage to Bourbaki, sparked a vigorous genre of self-parody among mathematicians.[43] It featured methods such as:

> THE METHOD OF INVERSIVE GEOMETRY. We place a *spherical* cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.[44]

In his autobiography, Schwartz recalled a version for the theory of distributions:

> Anything in the desert is a lion, but in the sense of distributions. For example, a stone is a lion in the sense of distributions. So it suffices to regularize it by a C^∞ function with compact support, and it becomes a real lion, which furthermore is infinitely differentiable![45]

Distributions offered mathematicians a sense in which every function was differentiable, just one step removed from its conventional interpretation. A ritual incantation of regularization by a smooth compact function completed the trick, yielding a genuine function whose difficulties of differentiation had been swept away in the calculus of distributions. Applied to lions rather than functions, a magical-seeming intervention became sheer nonsense.

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[41] Liliane Beaulieu, "Bourbaki's Art of Memory," *Osiris* 14 (1999): 219-251; Jean A. Dieudonné, "The Work of Nicholas [sic] Bourbaki," trans. Linda Bennison, *AMM* 77, no. 2 (1970): 134-145.

[42] Gerald L. Alexanderson and Dale H. Mugler, eds., *Lion Hunting & Other Mathematical Pursuits: A collection of mathematics, verse and stories by Ralph P. Boas, Jr.* (Washington, DC: Mathematical Association of America, 1995), 9-10, 26.

[43] Boas notes an influx of new methods after the article's appearance in Boas to Tukey, 5 Nov 1938, Series I, "Boas, Ralph Philip, 1938-1939 (Folder 1)," John W. Tukey Papers, American Philosophical Society, Philadelphia, PA.

[44] H. Pétard, "A Contribution to the Mathematical Theory of Big Game Hunting," *AMM* 45, no. 7 (1938): 446-447, on 446.

[45] MGC, 247. C^∞ functions were those that were guaranteed to have no irregularities, and so always had conventionally valid differential equations.

Schwartz's 1997 account of "A lion in the sense of distributions" had enough obvious misrecollections that I had included it in this essay just as an apocryphal illustration, true "in spirit" if not verifiably in fact. Then, while in the archives pursuing a different project, I came across a 1951 letter that confirms the story and its implications better than I could have imagined. In it, Boas reported that he was "collecting new methods of lion hunting with a view to bring Pétard's minorem opus up to date," and suggested "the office of Navel Research"—parodying the Office of Naval Research that emerged after World War II as a leading funder of American mathematical research and publication—may have sponsored the new project. Tucked between a method referring to Schwartz's fellow 1950 Fields Medalist Atle Selberg and a method deploying "a high speed computer," one finds:

> 2. The Schwartz method. We can catch <u>all</u> lions. Of course the one we get may be only a distribution, not a real live lion, but par abus de langage we may regard it as a lion.

Boas signed the letter "Yours in a cage."[46] Here, functions are replaced by lions and differentiating the former becomes catching the latter. Schwartz's theory let one differentiate any function, but the outcome was only guaranteed in general to be a distribution, equivalent to a function if the original function were differentiable. An abuse of language (whose Bourbaki connotations Boas signaled by leaving the phrase in French), however, let one carry on as though nothing was amiss.

"Lion Hunting" was an effective parody because this kind of "abuse of language" could be found in virtually any kind of recent mathematics. Schwartz promoted his theory in a context where mathematicians plied familiar and novel formulations in ways that made sense only when accorded a proper interpretive framework, and where it was not just common but often necessary for mathematicians to move between senses of a term in their arguments. The parody lampooned a modern mathematics that appeared to many like an exercise in making sense from licensed puns.

[FIRST LEVEL HEADING] MULTIPLYING AUDIENCES

The slippage licensed by Schwartz's language of senses appeared to some to be particularly pronounced among physicists. An early student of Schwartz's recounted that "a good many limited their knowledge of the theory of distributions to the ritual phrase 'according to Schwartz, it has a sense.'"[47] Schwartz himself did little to discourage this attitude, and relished the spotlight that came with his designation as physicists' redeemer. Even those who might have wished for a detailed technical justification rather than the personal guarantee of the charismatic French theorist would have found it hard to come by, especially through the written literature: before his 1950-1951 textbook, the bulk of the theory's formal development remained a promissory note alone. In the interim, Schwartz cultivated an image of a theory that offered an easy rectification to the problems of differentiation.

This image took the fore in his first major public lecture explicitly aimed at non-mathematicians, on December 4, 1946, at the Society of Radio-electricians, subsequently

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[46] Boas to Tukey, 5 Oct 1951, Series I, "Boas, Ralph Philip, 1941-1992 (Folder 3)," Tukey Papers (ref. 43). In addition, the letter includes a method "not yet fully developed" using paradoxical properties of "the Tomonaga-Schwinger theory" discussed, e.g., in Kaiser, *Drawing Theories* (ref. 13), and a method inspired by Bourbaki collaborator André Weil's wartime proof of the Riemann Hypothesis over finite fields.

[47] Bernard Malgrange, "Laurent Schwartz et la théorie des distributions," in Anné et al., *Supplément* (ref. 18), 67-74, on 67-68.

published in the *Annales des Télécommunications*.[48] Uniquely among his earliest presentations of the theory, Schwartz did not define the derivative of a distribution by analogy to integration by parts. Instead, he began with the difference quotient defining derivatives for conventional functions in one variable. Interpreting this in terms of translations, Schwartz constructed a corresponding difference quotient for linear functionals and manipulated that into the expression $T'(\varphi) = -T(\varphi')$. Only as an afterthought did Schwartz add that "the formula is, in fact, that of integration by parts."

Integral calculus was an effectively universal element of undergraduate science and engineering curricula in this period, and integration by parts was a core part of mathematicians' stock-in-trade.[49] While the telecommunications audience would surely have been familiar with integration by parts, Schwartz did not assume its use and teaching to be so central to their routine practice, and so the metaphor took a back seat. Rather, Schwartz stressed how distributions allowed "the complete justification of certain processes" where "one boldly uses the 'Dirac function'" and asserted that "the systematic manipulation of distributions permits evaluation with greater ease and fewer chances to commit errors." Despite its "complicated appearance," he proposed, his framework "is in reality very simple and demands just a little mathematical knowledge."

Claiming his theory's simplicity, Schwartz walled off the often-difficult recent developments in the theory of functions and topological vector spaces that (at least in principle) underwrote his distributions' prized rigorous foundation. Instead, he emphasized distributions as a conventionally acceptable and ultimately simple technical method learnable by rote and sight—precisely the kind of method his audience already had in the Dirac and Heaviside calculi. Simple could also mean superfluous, and for many it was enough just to know that distributions justified what they were already doing. This early presentation would prove more important for Schwartz himself, helping him hone a view of his theory as simple, fundamental, and meaningful for non-mathematicians, whether or not non-mathematicians themselves endorsed the sentiment.[50]

Schwartz's next major venue, what Paumier has called the "springboard" for Schwartz and his theory, arose from the Rockefeller Foundation's postwar interventions to rebuild a European and trans-Atlantic scientific community.[51] As a result of the developments described up to this point, Schwartz was positioned both intellectually and institutionally for an international launch under the new terms of postwar international science and mathematics, associated here especially with Rockefeller philanthropy. Directed by Warren Weaver, the Foundation's Division of Natural Sciences partnered with the Centre National de la Recherche Scientifique to conduct international colloquia centered around specific topics of contemporary research. Weaver intended to maximize informal interaction and scientific contacts and to spread the benefits of foreign visitors and research investment to provincial universities so as to reorganize French science on an American model of

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[48] Laurent Schwartz, "Généralisation de la notion de fonction et de dérivation: Théorie des distributions," *Annales des Télécommunications* 3, no. 4 (1948) : 135-140.

[49] Luciana Zuccheri and Verena Zudini, "History of Teaching Calculus," in Alexander Karp and Gert Schubring, eds., *Handbook on the History of Mathematics Education* (New York: Springer, 2014), 493-513.

[50] See PLS, 150-154.

[51] PLS, ch. 3, esp. § 3.3; Anne-Sandrine Paumier, "Laurent Schwartz (1915-2002) et le colloque d'analyse harmonique de Nancy, 15-22 juin 1947," *Gazette des mathématiciens* 147 (2016): 39-51; BPL; John Krige, *American Hegemony and the Postwar Reconstruction of Science in Europe* (Cambridge: MIT Press, 2006), ch. 4; Doris Zallen, "The Rockefeller Foundation and French Research," *Cahiers pour l'histoire du CNRS* 5 (1989); Doris Zallen, "Louis Rapkine and the Restoration of French Science after the War," *French Historical Studies* 17, no. 1 (1991): 6-37.

regional, national, and international competition. Such imposed decentralization significantly accelerated Schwartz and his theory's international recognition, as the French mathematician was still years away from a firm foothold in the elite institutions of Paris that could otherwise afford him such a profile.

One of an initial pair of Rockefeller-CNRS colloquia in mathematics was a meeting on Harmonic Analysis in June, 1947, at the Faculty of Sciences of Nancy, where Schwartz had been appointed in 1945. The meeting's roster featured researchers at a range of career stages from Paris, Nancy, the United States, Scandinavia, Switzerland, and Britain. As a local junior colleague to the colloquium's primary hosts, Schwartz assisted with organizational correspondence and was able to give a presentation of his own and to circulate his "propaganda tract" on distributions among the materials shared with participants in advance of the meeting.[52]

Danish delegates Harald Bohr (the younger brother of physicist Niels, and a mathematician of considerable international stature) and his protégé Børge Jessen were so impressed with Schwartz's performance that they invited him to lecture in Copenhagen the following October.[53] To Schwartz's "agreeable surprise," in preparation for the October visit Bohr had made a personal study of Schwartz's 1945 paper and turned it into a course at his university.[54] While Schwartz's presentation in Nancy favored the kind of difficult technical and conceptual considerations one would expect at such a specialist gathering, for his Copenhagen audience Bohr urged Schwartz to reemphasize the theory's potential as a bridge between pure and applied mathematics, in part in response to the institutional conditions of his Danish visit to both the university and polytechnical school. This framing—of a pure theory that redeemed and even improved technical practice—became central for Bohr. Fully in the thrall of Schwartz's visit, the Danish mathematical elder statesman committed to travel to North America early in the following year to lecture on topics including "the generalization of the notion of differentiability of Laurent Schwartz."[55] Bohr's lectures in Copenhagen and then New York on Schwartz's theory were likely the first such expositions outside of France and Western Europe, respectively. Indeed, I have argued in detail elsewhere that Bohr helped precipitate nearly every one of Schwartz's most important international milestones between 1947 and 1950.[56]

An early hint of Schwartz's formulation, however, had already in 1947 reached most American sites of mathematical research (and many beyond the United States) through a brief entry on Schwartz's 1945 article in *Mathematical Reviews*. Then less than a decade old, the review journal had been created by the American Mathematical Society with the financial support of the Rockefeller Foundation and Carnegie Corporation in response to the nazification of what had previously been the leading review journal in mathematics, the German *Zentralblatt für Mathematik*.[57] A short, ambivalent review of Schwartz's article

[52] Paumier, "Nancy" (ref. 51), 42-43.
[53] BPL, §5. The colloquium also helped precipitate trips to London and Oxford in 1948. MGC, 302.
[54] Laurent to Marie-Hélène Schwartz, 5 Nov 1947, in Schwartz, "Premiers travaux" (ref. 3), 114.
[55] New York University Archives, Papers of Richard Courant, box 30 folder 15, Bohr to Courant, 1 Nov 1947, and Courant to Bohr, 27 Oct 1947. Peter Lax (interview with the author, Mar 2014) recalls Bohr introducing Courant's group to distributions on that visit.
[56] BPL.
[57] See Reinhard Siegmund-Schultze, "'Scientific Control' in Mathematical Reviewing and German-U.S.-American Relations between the Two World Wars," *HM* 21 (1994): 306-329. The key philanthropic records are folders 1549-1560, Record Group 1.1, series 200, Rockefeller Foundation Archives, Rockefeller Archive Center, Sleepy Hollow, NY; and Series III.A. Grant Files, box 36, folder 10, Carnegie Corporation of New York Records, Rare Book and Manuscript Library, Columbia University Libraries, New York, NY.

signaled that he had proposed new definitions for "functions" and "derivatives" in order to justify the Dirac function and its derivatives, but the reviewer did not mention the term "distributions."[58]

It would take another two years for the CNRS to publish the proceedings of the Nancy colloquium.[59] However Schwartz published a revised and somewhat elaborated version of his contribution, aimed at specialists in mathematical analysis, in the *Annales de l'université de Grenoble* later in 1947.[60] The same year the CNRS proceedings appeared, the 1947 article was reviewed in both *Mathematical Reviews* and the *Zentralblatt für Mathematik*, the latter of which had been revived in Berlin in 1947, with operations in both Western and Soviet zones. For *Mathematical Reviews*, Irving Segal of the University of Chicago briefly noted Schwartz's new definitions and some of their claimed applications.[61] For the *Zentralblatt*, meanwhile, Freiburg mathematician Gustav Doetsch offered an exceptionally long and detailed review (running to some two-and-a-half pages of small print) praising both Schwartz's 1947 and 1945 papers as offering a "new way" that made difficulties surrounding the Dirac function, for instance, "disappear completely."[62] Doetsch, who directed wartime mathematical research from his post in the German Aviation Ministry, was at the time he reviewed Schwartz's article still suspended from the University of Freiburg under the institution's denazification regime.[63] Though the documentary record on Doetsch's Nazism is unclear, there was no doubt of his zeal for mathematical applications, and he became one of Schwartz's most ardent early non-francophone promoters. His reviews gave yet another layer of meaning to Schwartz's postwar project of sense-giving, one that must have been especially significant for an expurgated scholar in occupied Germany: according to Doetsch, distributions "gave citizenship" to the Dirac function and other deracinated denizens of a mathematics in need of new foundations.

[FIRST LEVEL HEADING] A CONSIDERABLE SIMPLIFICATION

On the strength of Bohr's endorsement, Schwartz was among four foreign mathematicians invited to headline the second quadrennial meeting of the Canadian Mathematical Congress in Vancouver and its associated summer seminar in 1949.[64] Here, the Dirac function's redeemer met its most famous user, English mathematical physicist Paul Dirac himself, who found Schwartz's lectures at least sufficiently engaging to remain awake for their duration.[65] In addition to Schwartz and Dirac, Polish mathematician Antoni Zygmund of the University of Chicago and Indian mathematical physicist Homi J. Bhabha of

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[58] H. Pollard, MR0018259 (8,264e), 1947, accessed on MathSciNet.

[59] Laurent Schwartz, "Théorie des distributions et transformation de Fourier," in Colloques Internationaux du Centre National de la Recherche Scientifique, *Analyse Harmonique: Nancy, 15-22 Juin 1947* (Paris: CNRS/Gauthier-Villars, 1949).

[60] Laurent Schwartz, "Théorie des distributions et transformation de Fourier," *Annales de l'université de Grenoble* 23 (1947): 7-24. The article was received by the *Annales* on 3 Sep 1947. The extent of verbatim correspondence and the nature and location of the changes between the CNRS and *Annales* versions of the article are consistent with the *Annales* article being a revised version of the CNRS one.

[61] I.E. Segal, MR0025615 (10,36a), 1949, accessed on MathSciNet.

[62] G. Doetsch, Zbl 0030.12601, *Zentralblatt für Mathematik* 30 (3 Jan 1949), 126-129, accessed on zbMATH.

[63] Volker R. Remmert, "Mathematicians at War, Power Struggles in Nazi Germany's Mathematical Community: Gustav Doetsch and Wilhelm Süss," *Revue d'histoire des mathématiques* 5 (1999): 7-59, on 37-43, 49. Doetsch was reinstated as professor in 1951.

[64] MGC, 305-306. *Comptes Rendus du Deuxième Congrès Canadien de Mathématiques / Proceedings of the Second Canadian Mathematical Congress, Vancouver, 1949* (Toronto: University of Toronto Press, 1951). I present the indirect but substantial evidence of Bohr's role in the invitation in BPL, §7.

[65] MGC, 309.

the Tata Institute of Fundamental Research rounded out the meeting's "Lecture Series on Research Topics."[66] Schwartz's eight summer seminar lectures closely followed his 1948 *Annales des Télécommunications* article, an English translation of which was appended to the typed notes distributed to participants.[67] His lecture notes took the article's claims about the theory's ease to an even greater extreme: distributions were now suited "for young students and engineers," a "considerable simplification, without requiring great mathematical insight."[68] Indeed, borrowing another Bourbaki phrase, Schwartz went so far as to assert that "it is possible to start the theory from zero."[69]

This latter version of Schwartz's theory, one emphasizing its metaphors and the formal manipulations they licensed, took center stage the following year on Schwartz's second trans-Atlantic voyage, to Cambridge, Massachusetts. At the opening ceremony of the 1950 International Congress of Mathematicians, he was one of two mathematicians to be presented with a Fields Medal by the selection committee's chair, Harald Bohr.[70] Though Bohr lauded the "sensational character" of the work of the first medalist, Atle Selberg, it would have been clear to those assembled that the balance of Bohr's enthusiasm lay with Schwartz and his theory of distributions. Indeed, I have elsewhere analyzed new archival evidence that makes clear that Bohr's enthusiasm for the mathematician and his theory led him to manipulate the Fields Medal selection itself to ensure that Schwartz would be a recipient.[71] In a presentation double the length devoted to Selberg, Bohr attested to his "considerable amount of pleasant excitement, on seeing the wonderful harmony of the whole structure of the calculus to which the theory leads and on understanding how essential an advance its application may mean to many parts of higher analysis."[72]

By 1950, Bohr was a comparatively seasoned evangelist for the theory of distributions. With a tradesman's polish and a convert's zeal, he depicted a theory fit for the exertions of young researchers, offering precepts that "may find their place even in the more elementary courses of the calculus in universities and technical schools." For many in his audience, this was the first glimpse of the theory of distributions. The theory took hold after 1950 in part because, for many, a single glimpse was enough. Its most striking claims were "so easy to explain," professed Bohr, that he "cannot resist the temptation, notwithstanding the general solemn nature of this opening meeting, to go into some detail." Hewing closely to Schwartz's 1945 article, from which he had derived his Copenhagen course, Bohr began with the Dirac function and its derivatives and the series of generalizations that would tame them, culminating with linear functionals. He then followed Schwartz's presentation of differentiation by way of integration by parts for functions of one variable, stressing the formal similarities that allowed derivatives to be defined for all distributions. Bohr's irresistible details, however, remained at a level of simple formal manipulations, framings, and comparisons, free from most of the technical scaffolding and theoretical apparatus of mathematical analysis from which they were derived.

In his brief time at the podium, Bohr demonstrated that a meaningful appreciation of

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[66] *Proceedings* (ref. 64), x.
[67] Laurent Schwartz, *Theory of Distributions,* Courant Papers (ref. 55), box 79, folder 19.
[68] Schwartz, *Distributions* (ref. 67), 2.
[69] Schwartz, *Distributions* (ref. 67), 1. On "starting from zero" in Bourbaki, see Beaulieu, *Bourbaki* (ref. 2), I: 150.
[70] On the status of the Fields Medal in international mathematics in this period, see Michael J. Barany, "The Myth and the Medal," *Notices of the American Mathematical Society* 62, no. 1 (2015): 15-20, on 16-17; Michael J. Barany, "The Fields Medal should return to its roots," *Nature* 553 (18 Jan 2018): 271-273.
[71] BPL, §8.
[72] Bohr, "Address" (ref. 4).

Schwartz's theory did not require anything approaching a technical mastery of the theory or its contexts in modern analysis. Bohr's irresistible details were, to be sure, closely tied to the proofs, structures, problems, and practices of those who had made distributions an object of sustained study, and for whom such details guided intuitions and shaped narratives and explanations. But the theory's advanced research community, such as it was, depended on a significantly larger audience of those interested in the theory at different levels and for different purposes, some of whom would become experts but most of whom would not. This large group of mathematicians newly familiar with distributions would supply, in the coming years, the theory's enthusiastic researchers, their supportive colleagues, seminar audiences, and prize committees. Without proofs and with very few formal statements, Bohr made Schwartz's theory available to its largest audience yet. It was metaphors and comparisons, not formal proofs and structures nor problems and practices, that constituted the theory for this group, at least at first.

The research community for the theory of distributions, importantly, did not include Bohr himself. While Bohr worked in areas of analysis related to those to which distributions were pertinent before his death in January, 1951, he has not typically been cited as a contributor to the theory itself. Because he lacked an identifiable research result about distributions, historians and mathematicians have almost entirely neglected his crucial contribution to the theory's meteoric early adoption—arguably his last major intervention on the stage of international mathematics. However that same lack was perhaps what made his ultimate contribution possible. Bohr had enough expertise to grasp and articulate the theory's significance, but without the messy experience from first-hand research of having to grapple with its intricate difficulties and ambiguities.[73] At a slight remove from the theory's sometimes dense technicalities and unresolved tensions, he was an ideal ambassador.

[FIRST LEVEL HEADING] A TRAVELING THEORY

The intercontinental fallout of Bohr's evangelism was swift and far-reaching. By the summer of 1950, the Paris publisher Hermann had released the first volume of Schwartz's inexpensively printed textbook on the theory of distributions, and the second followed in 1951.[74] Spurred by Bohr's advertisement, the textbook became a fixture of mathematics libraries and an obligatory citation for those adapting or applying Schwartz's theory. As they had for Schwartz's 1947 paper, Segal and Doetsch reviewed the volumes in, respectively, *Mathematical Reviews* and the *Zentralblatt für Mathematik*, ensuring a broad general awareness of the textbook in its relevant specialist communities.[75] These review journals also began, with increasing frequency after 1950, to signal researchers' burgeoning interest in distributions with reviews of new publications. These reviews, typically written in English or French by Schwartz's sympathizers (and sometimes by Schwartz himself), routinely referred

[73] Cf. note 9 on MacKenzie's "certainty trough" of trust in technical systems. One might cite Bochner (ref. 17) as someone technically 'closer' to distributions and correspondingly less confident in the theory. Likewise, Paul Dirac does not seem to have been especially impressed by the theory of distributions as an intervention in quantum physics after meeting Schwartz in Vancouver, and certainly falls to the left of Bohr on the corresponding certainty trough for the Dirac calculus.

[74] Laurent Schwartz, *Théorie des Distributions*, vol. 1 (Paris: Hermann, 1950). The work was published as part of two series which were often cited alongside or in place of the publisher: most commonly, the Actualités Scientifiques et Industrielles (no. 1091), and less commonly, the Publications de l'Institut de Mathématique de l'Université de Strasbourg (no. IX). The second volume, which contained the bibliography for both, was published by Hermann in 1951, and was no. 1122 in the Actualités series and no. X in the Strasbourg series.

[75] *Mathematical Reviews* accessed via MathSciNet, MR0035918 (vol. 1) and MR0041345 (vol. 2). *Zentralblatt* reviews from zbmath.org, Zbl 30.12601, 37.07301, 42.11405.

to Schwartz's textbook and terminology, even if the work under review was not framed as a contribution to Schwartz's theory by its own author.[76] The limited format of a review article proved an ideal mechanism for annexing and consolidating such wide-ranging works, with just enough space to signal broad uniting claims without presenting the details wherein lay the most significant differences and departures. To note a novel development in such a review often amounted to identifying a key term and connecting it to other terminological (often eponymic) touchstones of the field—a word-focused exercise in asserting the *sense* of the intervention. This lent the theory of distributions a self-reinforcing appearance of coherence and broad interest. Annual direct citations of Schwartz's textbook peaked in *Mathematical Reviews* at 15 (in 1955), and together its volumes were invoked in 52 separate *Mathematical Reviews* entries between 1951 and 1958.[77]

While Schwartz's textbook served for most mathematicians (including those who did not work primarily in French) as the definitive reference on the foundations of the theory of distributions, a range of introductory texts by others made the theory recognizable in a much broader range of settings, languages, and disciplinary idioms than would have been possible with Schwartz's writings alone. Geographically, these texts appeared in nearly every region with a significant mathematical publishing infrastructure, from South America to South Asia to the Soviet Union. Some authors aimed primarily to draw attention to Schwartz's own work.[78] Others, however, presumed that Schwartz's theory was in need of domestication or reframing for their particular audiences, particularly if that audience included physicists.

Canadian mathematician Israel Halperin, for instance, adapted Schwartz's Vancouver lectures into a short pamphlet aimed at mathematical physicists.[79] Halperin's pamphlet began with the Heaviside and Dirac functions and asserted that distributions gave "rigorous content and validity to the formulae of operational calculus" associated with those two figures and their functions.[80] U.S.-based Dutch mathematician Jacob Korevaar produced a series of articles "from the point of view of applied mathematics," also beginning with Dirac and Heaviside and their symbolic manipulation and claiming that Schwartz's distributions gave "a mathematical basis for these formulas and for the use of other improper functions."[81] Argentine mathematician Alberto González Domínguez, who likely first learned of the theory from Bohr's presentation at the 1950 International Congress, offered his own account just over a year later in Uruguay of distributions as a rigorous means of deriving formulas from quantum physics otherwise obtained symbolically—that is, of distributions as licensed

[76] E.g. MR0036949 (12,189d), MR0036950 (12,190a), MR0070969 (17,63c), MR0070970 (17,63d), MR0072937 (17,354c), MR0072938 (17,354d), MR0074491 (17,594e) accessed on MathSciNet.
[77] Citation figures are from MathSciNet. See also PLS, §2.3. *Mathematical Reviews* represents, of course, just a fraction of the textbook's citations across all publications in those years, during which the textbook was a more or less obligatory citation for works explicitly engaging Schwartz's theory.
[78] E.g. M.H. Stone, "The Algebraization of Harmonic Analysis," *The Mathematics Student* 17 (1949): 81-92, on 89-90 (among the earliest expositions of distributions in India); Marston Morse, "Trends in Analysis," *Journal of the Franklin Institute* 251, no. 1 (1951): 33-43, on 35-38.
[79] Israel Halperin, *Introduction to the Theory of Distributions* (Toronto: University of Toronto Press, 1952). The pamphlet was in sufficient demand to merit a second printing in 1960. Georgii Shilov, discussed below, supervised the pamphlet's translation into Russian in 1954: Israel Halperin, *Vvedenie v teoriiû obobshchennykh funktsii*, trans. M. S. Agranovich, ed. G. E. Shilov (Moscow: Izdatel'stvo Inostrannoi Literatury, 1954).
[80] Halperin, *Introduction* (ref. 79), 1.
[81] Jacob Korevaar, "Distributions defined from the point of view of applied mathematics," *Koninklijke Nederlandse Akademie van Wetenschappen. Indagationes Mathematicae* 17 (1955): 368-378, 379-389, 483-493, 494-503, 663-674.

wordplay.[82] Mathematical physicists themselves increasingly heeded Schwartz's early suggestions of distributions' relevance to their research, and in so doing found a theory in need of considerably more reformulation and adaptation than Schwartz had implied.[83]

Even pure mathematicians found many features of Schwartz's presentations wanting. Polish mathematician Jan Mikusiński advanced an operator-theoretic interpretation of the theory that stressed the notion of weak convergence, beginning with a series of articles in French from 1948 to 1950 and culminating with a 1957 textbook on the subject, with distributions exemplifying a more general phenomenon with wider potential applications than was evident from Schwartz's focus on mathematical physics and differential equations.[84] Schwartz, reviewing Mikusiński's work in English in *Mathematical Reviews*, explicitly compared the latter's approach to his 1945 exposition, and so claimed it for the theory of distributions.[85] Note the inversion: Mikusiński considered Schwartz's theory a narrow application of his own, and Schwartz likewise presented Mikusiński's theory as a specialized contribution to distributions. They sparred, in other words, over the *sense* of their respective theories.

While Mikusiński became a particularly widely-cited authority in Central and Eastern Europe, renderings like Schwartz's in compact sense-driven settings like *Mathematical Reviews* helped ensure that a great share of the Polish mathematician's citations in the West took his theory as Schwartz did. Oxford mathematician George Temple, for instance, claimed to have derived his "alternative and simplified exposition" of Schwartz's "highly abstract" theory of distributions from Mikusiński's, which he depicted as a means of domesticating distributions for "the physicist and engineer" by emphasizing mathematical techniques of approximation and convergence.[86] In contrast to the instrumental framing with which

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[82] Alberto González Domínguez, "Distribuciones y funciones analíticas," in *Symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Punta del Este 19-21 Diciembre 1951* (Montevideo: Centro de Cooperacion Cientifica de la UNESCO para America Latina).

[83] See Peters, "Distributionen" (ref. 19); A.S. Wightman, "How It Was Learned that Quantized Fields Are Operator-Valued Distributions," *Fortschritte der Physik* 4 (1996): 143-178. See also Heinz König, "Neue Begründung der Theorie der 'Distributionen' von L. Schwartz," *Mathematische Nachrichten* 9 (1953): 129-148 (see MR0055425); Werner Güttinger, "Quantum Field Theory in the Light of Distribution Analysis," *Physical Review* 89, no. 5 (1953): 1004-1019; J. Serpe, "Remarques sur l'application de la théorie des distributions à la théorie quantique des champs," *Physica* 20 (1954): 733-742.

[84] See Krystyna Skórnik, "Professor Jan Mikusiński—the 20th anniversary of his death," *Linear and nonlinear theory of generalized functions and its applications, Banach Center Publications* 88 (Warsaw: Institute of Mathematics, Polish Academy of Sciences, 2010), 17-29. Jan G. Mikusiński, "Sur la méthode de généralisation de M. Laurent Schwartz et sur la convergence faible," *Polska Akademia Nauk. Fundamenta Mathematicae* 35, no. 1 (1948): 235-239; Jan G. Mikusiński, "Sur les fondements du calcul opératoire," *Polska Akademia Nauk. Instytut Matematyczny. Studia Mathematica* 11 (1950): 41-70; Mikusiński, "Une nouvelle justification du calcul de Heaviside," *Atti della Accademia Nazionale dei Lincei. Memorie. Classe di Scienze Fisiche, Matematiche e Naturali. Sezione Ia. Matematica, Meccanica, Astronomia, Geodesia e Geofisica. Serie VIII* 2 (1950): 113-121; J. Mikusiński and R. Sikorski, "The elementary theory of distributions. I." *Polska Akademia Nauk. Instytut Matematyczny. Dissertationes Mathematicae. Rozprawy Matematyczne* 12 (1957), 54 pp.

[85] MR0036949 (12,189d), MR0036950 (12,190a) accessed on MathSciNet. The former article was received by *Studia Mathematica* in 1949 and was given this date in Schwartz's review.

[86] George Temple, "Theories and Applications of Generalized Functions," *Journal of the London Mathematical Society* 28, no. 2 (1953): 134-148; George Temple, "The Theory of Generalized Functions," *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* 228, no. 1173 (1955): 175-190.

Mikusiński's theory's crossed from East to West, Soviet mathematicians Israel Gelfand and Georgeii Shilov's monumental five-volume 1958 work *Generalized Functions* travelled largely intact (albeit much more slowly) with the help of systematic cover-to-cover translation programs that brought Russian works to English, French, and German audiences.[87] Here, the cultural and intellectual authority granted to both the Soviet authors and their institutional context helped them benefit from systematic efforts to comprehend foreign theories on their own terms.

Many important expository and programmatic works emerged from a proliferation of seminars and courses that mathematicians organized to teach and investigate the theory in its several guises. Postwar Paris witnessed an efflorescence of research seminars devoted to framing the latest developments of international mathematics through the concepts and styles of the seminars' participants.[88] In addition to his vigorous participation in such seminars, Schwartz offered a heavily-subscribed series of lectures in mathematical physics and other topics after moving to Paris from Nancy in 1953, and these were frequently an occasion to print inexpensive collections of lecture notes from Schwartz or his students.[89] These lectures helped secure Schwartz's reputation in France as a leading contributor to applied mathematics, despite his association with a Bourbaki program often seen as indifferent or hostile to applications.[90] Where Schwartz himself was not present, lecturers made more deliberate use of his textbook and their own contacts among researchers on distributions.[91] In the United States, Britain, and France, such courses prepared advanced undergraduates for further work in mathematics or mathematical physics by introducing a theory at once easily appreciated in its rudiments and notably on the discipline's cutting edge.

The same apparent features that made the theory of distributions an appealing subject for an advanced undergraduate course made it all the more promising as a topic of special lectures from visiting mathematicians. A former Trotskyist, Schwartz (and his American supporters) made a series of tactical mistakes during his application to visit the United States that nearly prevented him from coming in 1950 to accept his Fields Medal, and which created

[87] E.g. I.M. Gelfand and G.E. Shilov, *Obobshchennye funktsii i deistviya nad nimi*, Moscow: Gosudarstv. Izdat. Fiz.-Mat. Lit., 1958; *Verallgemeinerte Funktionen (Distributionen)* (Berlin: Deutscher Verlag der Wissenschaften, 1960); *Les Distributions*, tr. G. Rideau (Paris: Dunod, 1962); *Generalized Functions*, tr. Eugene Saletan (New York: Academic Press, 1964). Note that the titles of the translations reflect the francophone nomenclature of Schwartz, the anglophone nomenclature of Temple, and a combination in German.

[88] See Paumier, "Le séminaire" (ref. 19); PLS, ch. 5, which considers both Schwartz's seminar participation and the seminars' broader scope and background.

[89] Laurent Schwartz, *Théorie élémentaire des distributions* (Paris, 1955); see MR0080707; Martineau and Trèves, *Éléments de la Théorie des Espaces Vectoriels Topologiques et des Distributions*, fascicule I: *Éléments de la Théorie des Distributions* (Paris: Centre de Documentation Universitaire).

[90] P. Germain, "Applied Mathematics in France," *Proceedings of a Conference on Training in Applied Mathematics, Sponsored by the American Mathematical Society and the National Research Council under Contract NSF-C7 with the National Science Foundation,* Columbia University, 22-24 October, 1953, 48-53, on 53, in Series I, "American Mathematical Society. Committee on Applied Mathematics, 1948-1954," Tukey Papers (ref. 43).

[91] E.g. Rademacher Papers, box 10, folder 2, Rockefeller faculty archives, Rockefeller Archive Center; "Course on Distributions [1951-1952]," box 17, Papers of Marshall Stone, John Hay Library, Brown University (including an outline for Claude Chevalley's course on the theory at Columbia University); "Lectures in Applied Mathematics (Honours) Schedules 1955-56," folder C1, Lighthill papers, University College London, 26; M. J. Lighthill, *Introduction to Fourier Analysis and Generalised Functions* (Cambridge: Cambridge University Press, 1958). See DPM, ch. 5.

further difficulties for his further prospects for lecturing in the United States.[92] Mathematicians at farther-flung institutions, many of whom were present at the 1950 International Congress of Mathematicians to hear Bohr's presentation, capitalized on this obstacle to bring Schwartz to their own shores, making use of increasingly available scholarly travel funding from international bodies like the United Nations Educational, Scientific and Cultural Organization. Where Cold War politics might have been expected to limit Schwartz's travel, in many ways it had the opposite effect, especially for travel to what came to be known as the Third World (with crucial support from United States mathematicians and funding sources). After 1950, Schwartz lectured to enthusiastic audiences in a variety of venues over periods ranging from days to months, including trips to Yugoslavia in 1951, Brazil in 1952, Mexico and Tunisia in 1953, India in 1955, Colombia and Argentina in 1956, India again in 1957, and Argentina again in 1958.[93] In India, Colombia, and Argentina, his host institution arranged for lecture notes to be edited and published, and such notes circulated widely in both the centers and peripheries of elite mathematics.[94] These international contacts helped Schwartz and his supporters to establish a cohort of younger researchers reaching from France to Northern Europe, the United States, Brazil, and Argentina.[95]

To the extent scholars of distributions the world over formed and sustained a coherent research community, they did so through the means and media that brought Schwartz early recognition. Conferences and colloquia allowed them to couch new developments in familiar terms, often by letting recognizable symbolic manipulations stand in for an unwieldy mass of detailed research. New textbooks and expository works plied metaphors from local disciplinary idioms to domesticate an initially radical theory. Decorated experts followed the paths of Bohr and Schwartz, promoting and studying distributions in foreign lands. Most kept abreast of new work in the area through abstracting journals and related means of abbreviated communication. If mathematical research hinged on protracted engagement with fine points of difficult concepts, mathematical research communities were sustained principally with what could be shared quickly and recognizably, and could be readily assigned a sense-making sense. Despite genuine differences among theorists in Poland, France, Britain, the Americas, and beyond, the theory of distributions attained an intercontinental reach by accommodating the kinds of partially-shared legibilities that let each make sense of each other in their respective terms.

[FIRST LEVEL HEADING] METAPHORS AND LIVED INCOHERENCE

International research communities, much less intercontinental ones, do not arise spontaneously. They require interconnected but geographically dispersed groups of individuals to decide to study the same topic, secure the means to do so, communicate their

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[92] RG 65, FBI Headquarters Case File 100-HQ-367305 re Laurent Schwartz 490/45/01/4, box 5327, National Archives and Records Administration; DPM, ch. 4.

[93] See Anné et al., *Supplément* (ref. 18), 19; Laurent Schwartz, Mission Report, 1952, United Nations Educational, Scientific and Cultural Organization online document database (UNESDOC) 159434.

[94] Laurent Schwartz, *Lectures on Complex Analytic Manifolds* (Bombay: Tata Institute of Fundamental Research, 1955); Schwartz, *Ecuaciones Diferenciales Parciales Elipticas* (Bogota: Universidad Nacional de Colombia, 1956); Schwartz, *Lectures on Mixed Problems in Partial Differential Equations and Representations of Semi-Groups* (Bombay: Tata Institute of Fundamental Research, 1958); Schwartz, *Matematica y Fisica Cuantica* (Buenos Aires: Universidad de Buenos Aires, 1958).

[95] See Michael J. Barany, "Fellow Travelers and Travelling Fellows: The intercontinental shaping of modern mathematics in mid-twentieth century Latin America," *Historical Studies in the Natural Sciences* 46, no. 5 (2016): 669-709.

findings to each other, and recognize their work as part of a coherent body of scholarship. For the community that coalesced in the early postwar period around the theory of distributions, mathematicians met each of these requirements through means where evocative metaphors and suggestive framings did far more to foster interest and an appearance of common enterprise than did the rigorous technical constructs commonly associated with modern mathematics. That those metaphors and framings drew their power in part by helping distributions' advocates to promise rigorous foundations was, here, not an irony or a contradiction but a reflection of the changing nature of mathematical theory and research. Detailed, meticulous work by committed investigators on the theory's finer points would be important, to be sure, but that was not the work that made the theory travel so far and so fast, across languages and continents in the mid-twentieth century.

Integrating distributions into one's day-to-day mathematics required difficult learning and adaptation, but integrating them into one's worldview required little more than evocative metaphors like those of Schwartz's initial "propaganda tract" and Bohr's 1950 address—metaphors derived from quotidian practice but with a life of their own. These two promoted distributions by transferring the difficulty of mastering the theory to the lesser difficulty of appreciating it, of acting as though all one needed was a well-chosen rearrangement of symbols and a change in signs. Indeed, in this period this kind of wordplay-inflected appreciation was, in many settings, all one could hope for and all that was required. The rise of the theory of distributions bears witness to the outsized effect that short summaries in review journals, brief lectures by traveling scholars, and cursory conceptual introductions in regional publications could have for organizing researchers separated by distance, training, and idiom. These medium-dependent connections made it possible for the theory to move from the French provinces to multiple continents in less than a decade.

Mid-century contexts of communication fit some theories and mathematicians better than others, and ensured that the most communicable aspects of any given theory came to dominate how mathematicians understood and shared what they did. Distributions were distinctive in the extent to which the kinds of metaphors and wordplay that traveled so well in lectures and review journals were central features of the theory's earliest articulation. From Schwartz's first publication on the subject and reverberating through his and others' early expositions, the theory depended on deft reframings of familiar terms and operations like differentiation and integration by parts. As Schwartz, Bohr, and others rearticulated distributions in successive presentations in its early years, they emphasized and reemphasized the symbolic manipulations and claims (especially regarding the Dirac and Heaviside functions) that made the theory most recognizable, at the expense of the technical apparatus that they purported would guarantee the theory's rigor (and thus its merit as means of foundational rehabilitation). Distributions could thrive in part because such reframings had become, over the preceding half century, primary mechanisms of mathematical argument and exposition—ones based on the intermingling of Schwartz's two senses of "sense": having an (in principle) logical foundation and having a (working) context that promises such a foundation.

It was neither possible nor necessary that mathematical physicists in Manchester meant the same thing by generalized functions as operator theorists in Warsaw, function theorists in Chicago, or mathematical analysts in Rio de Janeiro. Distributions varied, in this period, not just along a spectrum from suggestive metaphors to detailed programs of research. Scholars' different articulations of the theory reflected the different frameworks and communities of the interlinked national and regional communities in which it was elaborated, debated, and pursued. In Brazil, Schwartz portrayed it as an ingredient in modernization—both of mathematics and of a nation. In France, the theory was strongly identified with its charismatic originator and (in many eyes) the abstract program of Bourbaki, allied with modernization of a different sort. In the United States, to which Schwartz could not travel with ease, the theory found a range of institutional homes and advocates. In England, Temple made it out to be a foreign theory whose utility and grace emerged only upon a suitable domestication. He understood the foundational significance of sequences and the topology of functions in a way that could appear radically different from, for instance, the understanding Mikusiński promulgated in Warsaw, but mutually recognizable formalisms, examples, and terminology let each develop his respective approach in his respective sense.

Divergent interpretations of the theory let scholars in new places with varied interests find something in the theory of distributions worth pursuing, and those same scholars followed their lines of inquiry to very different ends, enrolling other people and ideas along the way. Communities of distribution theorists and their respective interpretations of the theory came together, to the extent that they did, through plentiful but circumscribed opportunities to travel and through the circulation of publications and reviews. But most of all, they came together through metaphors. Though the theory took many forms in many places in its first decade, its users could deploy common analogies and images to convince themselves that they were studying the same thing, and to effect meaningful exchanges amongst themselves. Their conviction of common ground necessarily came before the long and hard work of reconciling diverse ideas and institutions.

Such rhetorical features tied together a community of researchers that, though highly differentiated, was itself integrated by parts. Among the chief challenges to the historiography of twentieth-century mathematics has been the technically difficult work of assessing the relationships between different theoretical productions. The early history of distributions offers a reminder that such technical reconciliation was an ongoing and difficult task for historical mathematicians themselves—one in which, moreover, mathematicians needed not always be wholly successful in order to sustain significant programs of research. Rather than seeking coherence in historical theories, historians of modern mathematics can heed what was often the lived incoherence of those theories and the necessarily superficial but nonetheless important ways such theories cohered nonetheless.

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