Proof and its Putting: Mathematics, Rigor, and Testimony

Michael Barany

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Abstract

Beginning with the question of how mathematics works, I explore the rhetorical, philosophical, and social elements which give surety and significance to mathematical proofs. I approach the question through the concept of witnessing, asking what it means both to bear witness and to testify, how those ideas might be applied to mathematics, and what consequences arise from a model of mathematics based on witnessing and testimony. To begin to answer these questions, I introduce such concepts as *sesquitextuality* and the *slightly scalene* example, and conclude that the special referential structure of mathematics, encoded in its many witnessing relations, is of great consequence to the ultimate functioning of mathematics as a discipline.

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Acknowledgements

This work is dedicated to Philippe Minard, who heard I was interested in science and philosophy and insisted that I read *Leviathan and the Air-Pump.* Philippe sent me toward a discipline which would ultimately prove tremendously accommodating to the project of theorizing mathematics which first sparked my interest at the Telluride Association Summer Program under the watchful eyes and invaluable guidance of Masha Raskolnikov and Gregory Tomso. Since my fateful freshman encounter with Hobbes and Boyle, I have had the benefit of many seminars with professors who have helped me to develop and expand my ideas. I would like to single out the following, in particular: Jonathan Culler, in whose seminar on Derrida and Literature I first experimented with the idea of witnessing in mathematics; Christine Leuenberger, for whose seminar on Knowledge and Society portions of chapter 6 were submitted; Naoki Sakai, whose ideas about translation have been a centerpiece for my thinking, and for whose seminar on Translation portions of chapter 5 were submitted; Suman Seth, whose mentorship and guidance has helped me in innumerably many ways, and whose seminar in Science, Race, and Colonialism was an extraordinary source of analysis on primary and secondary sources in the history of science; Marjorie Levinson, whose School of Criticism and Theory seminar led me to many of the structural theses in this work, especially those of chapter 4; and Sara Pritchard and Mike Lynch, whose seminar gave me the opportunity to explore more of the foundational literature in the sociology of science, and to synthesize some of the major theories of witnessing in science in a paper for the seminar which formed the basis of chapter 2.

Kevin Lambert was in many ways the unofficial advisor to this project when it started to come together two years ago. Over the past year, my main source of suggestions and criticisms has been my committee chair, Mike Lynch, to whose questions and insights this work owes a great deal. I would also like to thank my College Scholar advisor and committee member Tim Murray, who said he knew I could do it, provided I take another semester of French, and my mathematics major advisor and committee member Bob Strichartz, who has taught me an enormous amount, not only in mathematics, but about the discipline, history, and philosophy of mathematics, and who first passed me his copy of *Proofs and Refutations*, another major influence of this work. The dinner table at the Telluride House has been a constant source of crazy ideas, many of which have found their way into this thesis in a more thought-out form, and I thank my friends and housemates at Telluride for contributions as intellectual sounding-boards and sources of both encouragement and much needed distraction. Special thanks are due to Stephanie Kelly, by whose virtue my translation of Cauchy's introduction is comprehensible, and by whose constant support I was able even to undertake such a project as this. My parents Barbara and George, and sister Deborah, deserve special thanks for more reasons than I dare attempt to list.

I have been assured that there will always be a place for mathematicians in the academy, in no small part because it is the second cheapest department. All one really needs are a pencil, some paper, and a wastebasket. The cheapest department, of course, is Philosophy, for it doesn't need the wastebasket. Despite its length, I should hope this work in the philosophy of mathematics has shown some signs of restraint. And despite its disciplinary origins, this project did not come together without some financial support. Research grants from the College of Arts and Sciences Dean's Scholars and Undergraduate Research programs funded attendance at conferences and the School of Criticism and Theory. The latter was also assisted by matching funds from the SCT, and I am grateful to Dominic LaCapra and Jonathan Culler for helping me to attend the program.

Notes on Citations

This work draws from wide swath of critical and historical literature on a wide range of topics from a wide range of disciplines in order to address a specific question in a way which does not appear fit neatly in any of the established literatures. As a result, I have had to analogically appropriate many arguments by other authors. It should be clear to the reader in which citations the author is actually writing about mathematics; in cases other than these, the citation usually refers to a comparable idea of the author's applied to a different conceptual system.

When quoting French language sources, translations will be my own unless otherwise noted.

All emphasis in quotations will be the quoted authors'.

Chapter 1

Introduction

If there's any one thing we know about modern mathematics, it is that it works. This much is beyond dispute. Mathematics works for the thousands of mathematicians, both professional and professorial, who make it the explicit subject of their daily work. Mathematics works for the still greater number of people whose work draws heavily and explicitly from mathematics—be they scientists or engineers, accountants or actuaries. Mathematics works even for those who don't see parts of their world in mathematical terms. It provides tools and techniques for practical computation and problem solving. It provides attitudes and intuitions for philosophizing and comprehending common and uncommon situations. Mathematics, as it is manifested in innumerably many forms and contexts in the lived experience of every person, is something which absolutely, positively, certainly works.

That mathematics works, while true, is not an interesting proposition. No matter how one chooses to define what it means for something to work, mathematics can be shown to work. Mathematics is remarkable in its persistent and pervasive ability to work, at every time and in every place. If I begin to sound hyperbolic in my description of mathematics, it is not with the intent of building it up in order to strike it down. The subject of these pages will be the proposition that mathematics works, but my question is not whether it works, for that answer is as plain as day. My question is, in my mind, a far more interesting one: how does mathematics work? How is it that mathematics, which, after all, is the aggregated product of millennia of fallible humans, still manages to work? And not only work. How is it that mathematics still manages to work so infallibly and so well, so generally and so profoundly?

The approach to the question of how mathematics works which I propose to use will also require me to draw in other interrogative pronouns: When does mathematics work? Where does it work? For whom does it work? On what does it work? I will also have to ask, either explicitly or between the lines, the negative forms of these questions: When does mathematics fail to work? Where doesn't mathematics work? For whom does mathematics not work? To what is mathematics simply inapplicable? All of these questions shed different lights on the basic question of how mathematics works, or rather, how mathematics works at the times, in the places, and for the people when, where, and for whom it does work, and also how it functions even in contexts where it doesn't strictly work.

Most of the time, and by most people, this question at the heart of so many practices, both ordinary and extraordinary, which are integral to everyday life is never posed. To most people, most of the time, mathematics just works. Plain and simple. No explanation necessary. Mathematics works most of the time, and for most people, because it is beyond the need for explanation. Clearly, in theorizing mathematics I shall also find it necessary to ask what it means to theorize mathematics, and what the conditions of possibility are for such a theory.

Part and parcel with the great power of mathematics is its resistance to theorization. Where mathematics does admit theorization on a large scale, as is found in the discipline[1] of logic, it does so in almost purely mathematical terms. In these formulations, the question of how mathematics works is often reduced to mechanical questions of what propositions can be said to imply other propositions and under what circumstances. A more potent perspective might be found in the history and sociology of knowledge, and particularly of scientific knowledge. Studies in this analytical and critical tradition generally aim to confront taken-for-granted beliefs and knowledges in their own terms, attempting to unravel their means of articulation and reification.

And yet, there has been a long tradition of mathematical exceptionalism in a wide range of critical circles in and around science studies.[2] Mannheim famously exempted mathematics (and specifically the proposition $2 \times 2 = 4$) from his program in the sociology of knowledge, and most mathematical re-

[1]A *discipline* is defined by a set of methods and practices used to tackle various problems. A *field* is the set of such central problems and concepts to which the methods apply. It should be clear in context where I am preserving that distinction and where I use the terms somewhat interchangeably.

[2]Herrnstein Smith and Plotnitsky, 1997, p. 2.

sults appear utterly resistant to historicization, and at times seem entirely ahistorical.[3] For the early moderns, too, mathematical or geometric knowledge was something epistemically apart.[4]

To theorize mathematics, then, would be in large part to break down its edifice of exceptionality. I do not aim to claim that mathematics works just like everything else, for clearly it doesn't. Instead, I hope to show that mathematics, like everything else, is built on the lived work of countless social relations, and that it is the working of these dynamic networks of knowledge and sociality that makes mathematics so spectacularly successful at what it aims to do. I aim to challenge commonly held assumptions about the foundations of mathematics, but my goal is not to see them crumble. Instead, such an analysis should illuminate just how great the power of human collectivity is, and what towering monuments to truth and reason human societies can create not only in spite of but because of each individual's social existence.

Mathematical rigor was the basis of a movement formed to correct for individual failings, and it worked. It didn't work by giving some non-social, non-human grounding from which humans could escape the shortcomings of their humanity. This is the commonly held view of mathematical rigor, and it really does not give the community of mathematicians enough credit. How mathematical rigor actually worked is far more fascinating. It worked by giving a social, human grounding by which humans could *work through* their shortcomings and *bracket out* those cases where such a working through was impossible. Mathematics is a human discipline: borne of humans and disciplining humans by humans.

My primary analytic framework will be that of witnessing in mathematics. For most people, witnessing, in its proper habitat, is a simple enough phenomenon. In juridical settings, or in spectacles of mass culture, the vocabulary of witnessing is comfortable and meaningful to most. The last half century has seen a great many scholarly works on witnessing in other, less obvious contexts. Witnessing has been extensively analyzed and elaborated in the context of literature, science, and a host of other settings. This line of scholarship has developed important connections which enrich both our

[3]Alexander, 2006, p. 680. Mannheim still allows for the possibility of differences of opinion within the mathematical community, which can in turn be used to ground a limited sort of sociology of knowledge for mathematics. The basis for Mannheim's exemption is that he saw some specific facts or truths, such as $2 \times 2 = 4$, as utterly beyond dispute.

[4]Shapin and Schaffer, 1985, pp. 23–44.

understanding of the individual context being studied and our general understanding of the phenomenon of witnessing.

Among the areas still waiting for an in-depth analysis through the lens of witnessing is mathematics, and the present work aspires to begin to redress what I see as a common gap in the theory of witnessing and in the theory of mathematics. Because of the limited body of extant work even allowing for some sort of witnessed and witnessable character in mathematics, it will first be necessary to elaborate what witnessing in mathematics might look like. I will approach this problem from two angles. The first will be to describe witnessing in a way which enriches its potential for analytic complexity, especially when describing the rational sciences.[5] I will need a robust and nuanced theory of witnessing in order to have any hope of getting it to speak with mathematics.

The second angle will be to characterize mathematics in terms and from a perspective which opens the discipline up to a reading through the concept of witnessing. Such a characterization will emphasize the ways in which the work of mathematics is a socially achieved, reified, and regulated one. Describing mathematics and witnessing in common social terms has the potential to provide a conceptual meeting point wherein these concepts can interact. My analysis of mathematics will draw from a range of analytic frameworks, each with its own advantages and disadvantages. Part of my goal is to indicate the analytic flexibility of witnessing analyses in the face of mathematics.

In order to set the stage for this project, chapter 2 presents a general theory of witnessing in science. The goal of this chapter is synthetic. I attempt therein to bring together a wide body of critical literature on the subject of witnessing in science in order to piece together a useful model of the phenomenon. The chapter begins by describing what I call the *observer model* of witnessing. It then proceeds to break down and reassemble this model into a framework for witnessing which is flexible enough to account for a much greater range of social and theoretical inputs.

Chapter 3 offers a historical case study through which to view the lived work of mathematical production. In this chapter, I describe a historical

[5]By rational sciences, I mean any discipline which would fashion its methods around the use of reason to draw conclusions and evaluate claims and results. This includes, as I shall show, the so-called Boylean natural sciences, as well as mathematics, but may at least partially exclude other areas in which witnessing remains an important concept, such as literature and certain aspects of the law.

transformation in the conceptual foundations of rigorous mathematics which took place over the eighteenth and nineteenth centuries. My specific focus is the early development of Augustin-Louis Cauchy's Course in Analysis at the École Royale Polytechnique from 1816 until the appearance of Cauchy's first textbook from the course, in 1821. I claim that Cauchy's rhetorical, social, and mathematical accomplishment in this period was to refigure algebra as a potential source of sound and right method. This refiguration, in turn, made possible the whole cascade of foundationalist re-presentations of mathematics in the ensuing century, and would fundamentally reshape how mathematicians thought about mathematics. My analysis of Cauchy draws first from historical and biographical sources, and then turns to his 1821 textbook.

Chapter 4 gives the first of three approaches to the problem of witnessing in mathematics. Engaging explicitly with the semiotic models elaborated by Brian Rotman, I look at how mathematics says what it *means to say*. Unpacking the network of putative statements in a mathematical demonstration, I describe how mathematics invokes a model of production which isn't quite what it claims to be. At the end of this chapter, I introduce the concept of *sesquitextuality*, which characterizes the form taken by this peculiar model of mathematical productivity.

The second approach, in chapter 5, describes mathematics in terms of its *structures* for saying what it means to say. I argue that mathematical meaning is created and certified through a process of translation. Framing it within a wide body of critical theories of translation, I present turn of the twentieth century logical messianism as a theory of the ordering of knowledge and language. To help in this presentation, I include a brief detour to characterize the marked orderings of a mathematical work, and introduce a way of diagramming these co-articulated orderings. I close the chapter by offering an interpretation of mathematics in terms of Sakai's theory of heterolingual address.

Finally, in chapter 6, I take the proofs which are the subject of the previous two chapters and ask how they are socially constituted as rigorous and meaningful presentations. Introducing concepts such as *slightly scalene* triangles, this chapter attempts to characterize mathematics as a trade in examples, models, and heuristics, as well as credibility, trust, and narrativity. In this, as in the other chapters, witnessing appears both by explicit citation and by structural analogy. The chapter ends with a discussion of two proofs of a single theorem from Cauchy's textbook, and describes how the presentation of these proofs elaborated a specific framework for the production of right mathematics.

I end with some speculation about the place of such analyses as the one I am attempting within both critical and mathematical circles. There is, I think, a great potential for work of this sort to substantially enrich critical studies of everything from science to literature. For mathematics, I'm a little more skeptical. While some framings of mathematics made possible by the present theorization may well bolster certain practices of mathematics and threaten others, ultimately the point of this project is not to enrich mathematics. After all, as I have maintained from the outset, we already know that mathematics works. What we do not know, and what I wish here to explore, is how.

Chapter 2

Witnessing in Science

Any attempt to characterize witnessing in science in the modern era[1] is always at least a little misguided. One would do just as well to characterize current events in newspapers, or numbers in mathematics. Science as we know it today is so intimately bound up in witnessing that describing the role of witnessing in science is often tantamount to describing the role of science in science. Put this way, the question of witnessing in science remains important, but it is clear that it cannot be reduced to a discussion of witnessing as a part, aspect, or character of science. Witnessing in science must be approached with an eye to the workings of science as science and witnessing as witnessing.

Another problem comes in the converse to this question: how to characterize witnessing outside of science. Again, one finds that just as science is bound up in witnessing, witnessing is bound up in science. In every arena, witnessing invokes a certain claim to scientificity, or at least a certain standing with respect to science. From courtrooms to classrooms, witnesses of all stripes are constantly compared, be it explicitly or between the lines, to their expert and scientific counterparts. Witnessing and science interpenetrate in such a way as to make any discussion of witnessing a discussion about science, and any discussion of scientific witnessing a discussion of witnessing in general. It is beyond the scope of the present work to explore the endless kinships between scientific witnessing and witnessing writ large. At present,

[1]My historical period of interest starts with the early moderns and their intellectual and philosophical influences, and continues through to the interwar and early post-war twentieth century. The main foci for most of this work, however, will be the turns of the nineteenth and twentieth centuries.

it will have to suffice to gesture at the edifice of witnessing through science, and hope that what is written here can be read in such a way as to shed light on the phenomenon of witnessing in general.

2.1 Playing the Witness

We begin with the question 'What is a witness?' It is a question at the heart of the Social Studies of Science literature. Indeed, a foundational work in the discipline argues that modern science itself was predicated on the establishment of a particular model and practice of witnessing.[2] To answer that question, we will have to draw implicitly and explicitly from a wide range of figurations of witness-hood. There is a juridical figuration: an eye witness was present for an event and must recount her phenomenal experience so that the truth of a claim may be established; an expert witness must testify on the basis of her scholarship and general experience as to how we are to interpret an eye witness account, or a piece of gathered evidence. There is a political figuration: a witness is someone who bears witness to a phenomenon, a moment, or an idea. There is a literary figuration: a witness is someone who changes the character of a circumstance by moving it from the unobserved to the observed. There is also a scientific figuration: a witness certifies matters of fact by attesting to her observations of the working of nature, under the constraints of the laboratory or in a more naturalistic setting.

In each of these figurations, a witness has many parts. She has a certain subject status, for even if the witness is not a person (the witness could be, for instance, a wall bearing witness to the fall of a great city, or a machine or scientific apparatus bearing witness to the minute changes in chemical composition of a reaction in solution) she must occupy the enunciative position of the subject in order to attest to having witnessed something. She has an associated narrative, called testimony,[3] through which she performs her status as witness. She has something to which she bears witness. Lastly, she has some system in which she bears witness which accords to her a degree of credibility and a right to play the part of witness.

All witnesses play the part of witness, and this is no mere figure of speech. I will take the position, no longer uncommon in a range of established litera-

[2]Shapin and Schaffer, 1985.

[3]We must be careful, however, not to reduce testimony to narrative. See Derrida, 2000, p. 38.

tures,[4] that witnessing is a phenomenon established through a social, political, rhetorical, and epistemic positioning made possible by a variously defined system of role playing and role performance. I will further hold that this sort of performance has a special relationship with the visual, and that this relationship operates both literally and metaphorically to enforce an ocular hegemony on the practice of witnessing. Indeed, the language of perception, which has been traced to sites as varied as Dutch descriptive painting[5] and French microbiology,[6] pervades the very terms by which we elaborate and comprehend knowledge.

Clearly, knowledge presents itself at eye level. It is easy to speculate why. Vision is at the literal and figurative center of our knowledge practices. Literally, it is our means of observing experiments and reading scientific articles, of surveying a landscape or a text. Figuratively, it is the etymological core of discursive commonplaces such as demonstration, insight, clarity, and comprehension. Even when the eye is distrusted, it remains the instrument of its own reinforcement and affirmation.[7] The problem becomes not what can be done to replace the untrustworthy eye but rather what can be done to synoptically supplement its faculties in such a way as to minimize its shortcomings while opening up its truth-granting power.

The relationship between vision and knowledge is both a natural and difficult one to consider because of its central position in many knowledge communities, and especially across the sciences. A basic starting point of ethnography, including the ethnographies of science exemplified by Bruno Latour's early work, is to begin by recording the flood of sensory, and in particular visual, data which come out of any space of production—cultural, scientific, or otherwise. Historical and cultural studies begin more often than not with textual sources which are annotated, condensed, and synoptically arranged to create new understandings. Even when interviews form the basis of sociological studies, it is the transcripts of these interviews which make up

[4] c.f. Shapin and Schaffer, 1985; Derrida, 2005.

[5] Shapin and Schaffer, 1985, p. 17

[6] Latour, 1999, p. 260.

[7] c.f. Descartes, 1641. Among historically dominant philosophies of science, it is common to distinguish between properties of objects which may reliably be discerned by observation and those about which observation may mislead. For Popper, it is important to distrust observations until one can rigorously test them, and so the scientific eye becomes the guarantor for the fallible eye. We might call the challenge of identifying when one set of observations is sufficiently objective to certify the preceding ones the *witness's regress*, in homage to Collins, 1985.

the sedimented material base of the knowledge work that follows. To take yet another example, studies in conversation analysis have developed increasingly nuanced ways of visually recording the many layers of information encoded in verbal exchanges.

I claim that this tactic of visuality goes hand in glove with another one found throughout the field of science studies: that of *playing the stranger*.[8] This methodological starting point can be found whether the subject is contemporary[9] or historical,[10] and seems independent of whether or not the study's subjects work with real or imagined objects.[11] The methodological preambles to these widely varied texts all read more or less the same. Let us take a characteristic example.

The opening chapter of Shapin and Schaffer's *Leviathan and the Air-pump* observes a tension between what they term "member's accounts" and "stranger's accounts."[12] The former suffer in what they don't ask, occluding most of the most productive and fruitful questions to be asked about the structures and practices one is studying.[13] The latter suffer in asking too many questions, taking nothing for granted and so having nothing to go on.[14] The trick is to *play* the stranger, strategically disbelieving specific taken-for-granted truisms. At the point of contact between the researchers and their subject, some resource must be found which betrays just enough information to begin to tackle meaningful questions, but not so much information as to unduly naturalize the researchers' subjects.

This information, though it takes many forms, is almost invariably visual. Historians take texts and tracts, or more specifically words, at face value as investigative resources and entry points into cultures of production. They grant highly invested terms the status of momentarily empty graphemes, purely visual phenomena, waiting to be filled by a clever cross comparison of texts from a contextually-bound contemporary controversy. Ethnographers of science, too, begin with the textual productions of scientific inscription, marveling half-credulously at the myriad meanings tucked into piles upon

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[8]Shapin and Schaffer, 1985. p. 6.

[9]Latour and Woolgar, 1986, p. 29.

[10]Shapin and Schaffer, 1985, p. 6.

[11]c.f. Rotman, 1988, p. 6.

[12]Shapin and Schaffer, 1985, p. 4.

[13]Ibid., pp. 4–5.

[14]Ibid., p. 6.

piles of meticulous scribbles.[15] Playing the stranger, it would seem, tends in the end to mean *playing the witness*, taking complex situated phenomena and divesting them of all meaning beyond what can be apprehended from a critical distance by a detached observer. Whether they intend it or not, these observers make their subjects into visual subjects.

But their tentative translation does not stop at this visual transmutation. The real task at hand is to make a story of these glimpses, to piece together a theory and a method from the chaotic madness of scientific machines of articulation. In storying these visual phenomena, our scientists of science *become* witnesses, testifying to the heterogeneous character of science itself. Our ethnographers and historians are not the first scientists to ground their studies in practices of witnessing. It has been argued that this distinction belongs to Robert Boyle, who made witnesses the center of his seventeenth century experimental program.[16] When historians and ethnographers play the witness, they participate in a system of knowledge-making invested with centuries of meanings and implications.

Thus, whatever one means to do when one investigates the status of witnessing, one will have to make room in the account, in or between the lines, for what we do when we witness, and in particular when we witness witnessing. A 'science studies' account of witnessing always stages a play within a play, and always invokes from within the question of what it means to play and to perform and to act as though we are more than actors. It is to un-write, underwrite, and annotate the script as it is written.

2.2 The Observer Model

Let us begin with the question of what the witness actually does. Here, I wish to consider witnessing in a broad sense, covering a range of deliberate and non-deliberate agencies in both people and non-people. I must be aggressive in this aspiration to generality, for witnessing itself is something which has ardently avoided compartmentalization. After all, Boyle took his notion of witnessing and testimony from a very narrow context in jurispru-

[15] Latour and Woolgar, 1986, and Latour, 1987, offer particularly dramatic examples of this rhetorical-methodological phenomenon.

[16] Shapin and Schaffer, 1985. Serjeantson, 1999, p. 208.

dence[17] and rhetoric,[18] and made it into the basis of facticity.[19] Moreover, I will treat witnessing and testimony as coextensive phenomena. This means that wherever I refer to witnessing or testimony one must necessarily infer the presence of the other, for testimony clearly testifies to something, and witnessing without testimony is mere observation, if even that.

We have already encountered a significant problem. Part and parcel with the intermixing of witnessing and testimony is the difficulty of determining where witnessing begins, and what its components are. As we shall see, witnessing brings into play an entire constellation of social relations pertaining to the status of, among other things, the witness, the putative event being witnessed, the objects of the witness's testimony, and the testimony's audience. I will attempt a detailed discussion of this problem as this work unfolds. As a point of departure, I will elaborate a particular model of witnessing which sets in play the different roles and relationships which will need to be examined and complicated. I call this model the *observer model* of witnessing.

Central to the observer model is, it should come as no surprise, the posited figure of the observer. The observer is taken as an independent subject, detached from the outset from both that to which she bears witness and those to whom she testifies. With the observer at the center, the observer model divides witnessing into a series of discrete and separable event-objects. The scene of witnessing begins with an event, or experiment, at center stage, with the observer (or observers, in the role of witness or witnesses) positioned at or near an Archimedean point off stage, from which she or they can see everything that takes place, albeit not necessarily from an a-perspectival or unobstructed standpoint.[20] The witness is then called to testify, in a literal or figurative courtroom, and her deposition translates in a clear and complete way all of the pertinent aspects of the witnessed event.[21] That such

[17]Shapin and Schaffer, 1985, p. 56.

[18]Serjeantson, 1999, p. 200.

[19]Shapin and Schaffer, 1985, p. 25.

[20]This position might be called the god-position, for it is the one supposed in the god-trick of "seeing from nowhere." See Haraway, 1999, pp. 176–177; and Daston, 1999, p. 111. A significant conceit of scientific witnessing is the possibility of removing even the limitations of perspective, so that the observer not only sees from *nowhere* but sees *everything* from nowhere. Ultimately, the observer must be both sufficiently present and sufficiently self-present (that is, conscious and aware) to be able to faithfully represent the event. Derrida, 2005, p. 79.

[21]Such testimony represents a moment of decision between testifying accurately, erring

a correspondence is impossible[22] needn't concern us here, for the observer has access to an endless expanse of time and an endless reserve of attention[23] and memory with which to fill in or clarify any details in which her testimony is felt to be lacking. That the observer is never actually able or in a position to access any of these qualities is not without significance, and I shall return later to the question of what it means for something to be accessible but impossible to access.

There are several invisible participants in the observer model to accompany the witness, who becomes visible at the site of testimony, and the event or experiment, which is visible all along. For one, the audience or public interrogating the witness is never seen in the observer model. Like Santa Claus, the observer model's audience sits in omniscient bliss at the North pole and tallies up accounts of events until it has reached a point of determination. In the end (which may or may not ever materialize), the good testimony (for it is possible that testimony is inaccurate, or, worse, deliberately deceptive) is presented with the status of factual truth, and the bad testimony is offered a lump of epistemological coal and its bearer is sent to bed without any credibility.

The last invisible participant is easy to miss. She is the theorist, epistemologist, or anthropologist who posits and thereby observes the scene of witnessing. The theorist cannot be ignored, for it is, after all, her theory which brings the participants into being, and it is the theorist who brings her theory into being. Ultimately, we will need to draw the theorist into her own theory as we attempt to account for the troubled and contingent relation between the many actors in her witnessing drama.

2.3 Witnesses, Good and Bad

When the observer model's audience regards the testimony of the observer, it faces the problem of sorting good testimony from bad testimony. This is, of course, the same problem as that of sorting good witnesses from bad

in good faith, and committing perjury, each of which must be possible within the testimony. Derrida, 2005, p. 78.

[22] Hanson, 1965, p. 25.

[23] Attention span is especially important if the experiment is controversial, as flagging attention is often what decides controversies or makes them disappear. MacKenzie, 1999, Accuracy, p. 350.

witnesses. The audience's analysis will typically proceed along two fronts: plausibility and credibility.[24] These criteria for the reliability of the witness and testimony, along with those for establishing the accessibility of the experiment or experience, are what ultimately allow witnessing to produce matters of fact.[25] The first, plausibility, pertains to the scene and context of observation, while the second, credibility, addresses the actual person of the observer.

Some historical context helps to explain the particular values implicated in Boylean, or scientific, witnessing, from which the observer model has extensively drawn. The seventeenth century saw a rapid rise in the theoretical importance of natural history as an avenue of thought and inquiry, largely due to the philosophical influence of Francis Bacon. This approach, emphasizing descriptive and observational analysis of the world, brought with it a parallel increase in the importance of testimony to prominent thinkers.[26] Testimony was sufficiently important that when an outsider was cited to verify a fact, he was likely to be referred to not as a judge but as a witness.[27] This model grew out of, but eventually diverged from, Renaissance theories of testimony which did not distinguish between testimony and authority.[28]

This identification, and in particular the attention paid to the credit and faithfulness of the witness, specifically invited an emphasis on gentlemanly culture which Boyle used as a resource in establishing the role of the witness in experimental truth-claims.[29] At the same time as testimony and authority began to diverge, the gentlemanly classes, which were conveniently enough also the scholarly classes, were growing increasingly suspicious of truth, certainty, rigor, precision, and, in particular, mathematics.[30] Instead, they built a system in which trust and credit were the central components of a well maintained cognitive order.[31]

The credibility question then became: how could one trust a particular witness? For the gentlemen-scholars, the answer was easy enough: one could

[24] Shapin and Schaffer, 1985, p. 39.
[25] Ibid., p. 336.
[26] Serjeantson, 1999, p. 208.
[27] Shapin and Schaffer, 1985, p. 68.
[28] Serjeantson, 1999, p. 204.
[29] Ibid., p. 205.
[30] Shapin, 1994, p. xxx. This attitude dominated epistemic thinking even among mathematicians through at least the eighteenth century according to Richards, 2006, p. 703.
[31] Shapin, 1994, p. xxv. Serjeantson, 1999, pp. 197, 201.

trust a witness if he had the moral constitution of one sufficiently noble and gentlemanly.[32] Indeed, prestigious witnesses were centrally important to Boyle's experimental program.[33] Honest and upstanding gentlemen had no reason to misrepresent their experiences or distort the truth because they were both virtuous and free of material and social want or need, and thus capable of free action.[34] The problem of trust was transformed into into one of identifying gentlemen.[35]

Free action was crucial, but only in highly regimented manifestations. Boyle needed to constantly police his witnesses to avoid a level of individualism which would jeopardize the stability of their assent.[36] Honor could be invoked as a sort of currency for gentlemanliness, and thus one who gave bad testimony could be inferred to be otherwise dishonorable and made subject to social rebuke.[37] The possibility of being discredited was indispensable. Witnesses needed to be free to uphold or fail to uphold their honor in the process of testimony. It was for this reason that mathematics and its associated reasoning was especially despised, for they compel assent in a way which denies the possibility of volitional action.[38]

One might ask here just what it means to compel assent. On its surface, compelled assent is simple: it is a demand for assent which does not appeal to the volition of those to whom the demand is made. In ordinary testimony, particularly within the observer model, the witness presents a narrative in which she asks the audience to believe her. The witness's relationship with the non-witness is one based on privileged access to the witnessed event, as well as the credulity which allows such access to be testimonially transfered to the non-witness. Compelled assent short circuits this relationship. It does so by displacing the problem of credulity and dispersing the means of access to the event. In mathematics a prover can appeal to the self-evidence of the steps of a proof or the justification of a proposition to give the

[32]Shapin and Schaffer, 1985, p. 130.

[33]Ibid., pp. 57–58.

[34]Shapin, 1994, pp. xxvi–xxvii.

[35]Ibid., p. xxvi.

[36]Shapin and Schaffer, 1985, 56.

[37]Shapin, 1994, p. xxvii.

[38]Shapin, 1988, pp. 32, 50. Shapin and Schaffer, 1985, pp. 101, 143. Voluntarism plays a key role in Descartes's mathematics, as well. Funkenstein, 1980, pp. 181–182. In a more contemporary context, the compelled and systematic production of assent has been called into question by practices of digital computing in mathematics. Rotman, 2003. See also MacKenzie, 1999.

audience primary access to the mathematical truth therein and so obviate the need for the prover. Other forms of presentation and justification are found throughout the natural sciences.

Thus, though the testimonial structure of scientific communication may ask one to believe in the testimony, this request is always bracketed by an epistemological compulsion which dusts aside the possibility of dissent. As the conclusions of a scientific argument are presented, it cannot be said that the scientist genuinely asks one to believe her.[39] Belief and assent, for Derrida's scientists, is to a certain extent a foregone conclusion.

There remains, however, a purpose to the testimonial form and the theatrics of requesting assent. Derrida writes that the injunction to believe is a pragmatic and performative one, not just the sort of theoretical invocation it is often assumed to be.[40] Assenting to a scientific demonstration presents itself as a volitional action, even as it certifies itself as independent of such volition. This performance of volition, which is certainly evident in Boyle's gentlemen-scientists, must always make possible and be predicated on the pre-given certainty of scientific testimony.

Whenever knowledge is to be produced, and particularly whenever new knowledge is to be produced, this model of assent might be called into question. After all, it is implausible that something which has never before been observed could be assured of an unchallenged reception. Indeed, testimony always involves a certain testifying to matters of miracle.[41] Any scientist will attest to the element of the miraculous in her work, and in the discovery and invention of new things. To scientifically report new knowledge is to testify to the constitutionally unheard-of and miraculous. That the air should have a spring to it which was capable of cohering two marble plates was certainly not an established truth which every school child could recite. Boyle's narrative trick was to naturalize his observations in such a way as to make miracles part and parcel with the functioning of nature and the domain of natural philosophy. In this way, the miraculous could also be the foregone, for both nature and miracles could be seen as external things which could be

[39] Derrida, 2005, p. 77. Derrida extends his argument to everyone from historians, to scientists, to mathematicians. Such a general characterization of testimony can be read into virtually any area of knowledge work, and so we must be careful not to exempt science or non-science from analyses of each other's respective practices, while still respecting the marked differences between them.

[40] Ibid., p. 76.

[41] Derrida, 2000, p. 75.

mastered by observation and right method.[42]

The sense of testimony was substantially transformed in Boyle's time by the divergence of testimony and authority. In the early modern period, testimony changed from being an argument itself to a being a source of facts which could be used in arguments which would follow.[43] Common prudential wisdom became systematized into a means of evaluating testimony on a wide range of axes, including multiplicity, plausibility, directness, and knowledgeability.[44] Multiplicity was of particular importance to Boyle, who multiplied his witnesses in many ways.[45] He used his multiplicity of witnesses explicitly in his debates with rival natural philosophers.[46] Some witnesses in the pack were more important than others, and those who were especially prestigious or especially inanimate[47] had the most authority.[48]

Plausibility was created primarily through the use of right method. It was imperative both that experiments be done, and that they be done correctly.[49] They should be done, moreover, with reference to explicit procedures which obviate the need for expert judgment.[50] Increased plausibility in testimonial accounts could be obtained through verisimilitudinous narratives laden with excess detail which included reports of failed as well as successful experiments.[51] In Boyle's case, there is a clear sense that he dramatized having performed his experiments.

Buried in these negotiations over who was qualified to witness was the

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[42]External nature should be compared to the *internal* universal signifier of Derrida, 1974, p. 12. Such a universal signifier is in some models the very condition of truth. See chapter 5.

[43]Serjeantson, 1999, p. 226.

[44]Shapin, 1994, p. xxix.

[45]Shapin and Schaffer, 1985, pp. 25, 59.

[46]Ibid., p. 217. Multiplicity also led to some specific criticisms from Hobbes, including the difficulty of simultaneous witnessing, the difficulty of reconciling multiple sources of testimony, and the epistemological question of why so many witnesses should be necessary in the first place. Ibid., 114.

[47]Not all of Boyle's allies were human. In the Boylean scientific tradition, especially as elaborated by Latour, experimental apparatuses are made into witnesses through processes of inscription, which allow stimuli observed by the apparatus to be recorded in a useable way.

[48]Ibid., p. 218.

[49]Ibid., p. 62.

[50]Such a strategy of impersonality is a common response in disciplines or practices facing strong outside pressures and challenges to their credibility. See Porter, 1995, p. xi.

[51]Shapin and Schaffer, 1985, pp. 63–65.

more fundamental question of whether witnessing, and in particular the human senses, could produce any meaningful knowledge of nature. Boyle's chief opponent, Thomas Hobbes, for one, argued that common sensations were an untrustworthy guide to the natural world.[52] Observation could not be epistemologically privileged for Hobbes because he judged there to be a weak correspondence between sensory impressions and the external objects which provoke them by impinging on the sensory organs.[53] This dispute over the reliability of the eye and other sense organs was a central part of the debate between the two philosophers.[54] The use of testimony and evidence was part of broader contemporary debates as well, including those over the status of miracles and those who claimed to have witnessed them.[55]

Boyle and his allies were able to counter Hobbes principally through arguments derived from the credibility and plausibility of the testimony itself. In an almost literal fashion, witnessing and testimony collect and aggregate sensory experience, gathering together agreement over the true state of nature.[56] By this virtue, substances and events which may exist only locally come to have a universal character.[57] On an individual level, direct experiences were widely touted as the surest sources of truth.[58] Instruments could also be used to augment the senses and increase their reliability.[59] A century later, d'Alembert returned to math itself as the only hope of make imperfect sense testimony reliable.[60]

Observations themselves could be insulated from criticism by distancing them from their interpretations. With the right rhetorical tactics, disagreements about observations can be shifted from questions regarding the reliability of sense perceptions to arguments over the practices and assump-

[52]Shapin and Schaffer, 1985, p. 84. We can compare this position to that of Cartesian doubt in Descartes's *Meditations on First Philosophy*, which formed a basis of Edmund Husserl's influential phenomenology. Descartes, 1641. Like Husserl, Descartes' turn to philosophy came after a brief flirtation with a project to universalize mathematics. Schuster, 1980, p. 41.

[53]Shapin and Schaffer, 1985, p. 102.

[54]Ibid., p. 18.

[55]Serjeantson, 1999, p. 196.

[56]Shapin and Schaffer, 1985, p. 152.

[57]Latour and Woolgar, 1986, pp. 65–66.

[58]Shapin, 1994, p. xxv.

[59]Shapin and Schaffer, 1985, p. 36. See also the footnote on page 24.

[60]Lambert, Moral Fibers, pp. 25–26.

tions going into the observation process.[61] This shift, called externalization, invests data with increasingly specific meanings and interpretations drawn from experimental design and analysis.[62] A successful challenge to a matter of testimony then has the effect of denying higher levels of meaning to an observation, while keeping the observation itself intact.[63] Boyle was particularly successful in insulating experimentally produced matters of fact from criticism by policing the boundaries separating facticity from philosophy.[64] A final level of insulation protects nature from observation itself by refusing to allow speculation on the mechanical foundations of natural phenomena.[65]

The structures described above appear more than just metaphorically or by allusion in other settings of witnessing, including literature and the law. It is part of the normal course of intellectual history for conceptual categories from different modes of witnessing to interpenetrate and inform each other.[66] Boyle drew extensively from the rhetorical and juridical models of his time,[67] and rhetoric and law have since appropriated scientific terms and models. Untrustworthy observers are a commonplace in the courts, and untrustworthy narrators have a long pedigree in literature and literary criticism. In any system where there are rival accounts produced by rival agents, which is to say in any system at all, competing credibilities always invoke figures of good and bad witnesses, good and bad agents, and good and bad stories. These good and bad stories, without which good and bad witnesses would be unthinkable, or at least indiscernible, are the subject of the next section.

2.4 Testimony

The witness's work comes into its own in the form of testimony, the specific mode of rhetorical production where the identities of the witness and the

[61] Pinch, 1985, p. 8.

[62] Ibid., pp. 9–10.

[63] Ibid., pp. 15–16.

[64] Shapin and Schaffer, 1985, pp. 24–25, 42.

[65] Ibid., p. 24. For instance, the clock metaphor for nature posits that there are many different mechanical foundations which could produce teh same natural phenomena, just as there are many different ways to arrange gears behind a clock to produce the same effect.

[66] See Shapiro, 1986.

[67] Serjeantson, 1999. For a sense of the broad extent to which the juridical analogy could be applied, see Latour, 1993, p. 23.

witnessed are forged. The juridical model of testimony provides the simplest means for conceptualizing the act of testifying, especially in the observer model. The observer is called forward by an agent of the law and asked to translate her observations into spoken statements, which are then recorded in writing as part of the official transcript of the trial. For the early moderns, the juridical analogy was important to scientific testimony, but it was not the only framework to play a central role in the constitution of the modern witness.

Testimony, as a form still not wholly divorced from argument,[68] was included within the realm of logic,[69] and was thought of as a written form as much as a verbal one.[70] Within logic, testimony had the role of an 'inartificial argument,' or proof, and was imagined as discursively and conceptually pre-fabricated.[71] This sense of already being there, of not deriving from the witness, allows testimony to become invisible within a wide range of intellectual practices.[72] Invisibilized testimony hides all but the putative event or experiment itself, making the observation speak for itself in an account which is unaccountable to anyone or anything.[73]

The autonomous speech of the experiment naturalizes its phenomena at several levels. First, its linguistic-descriptive modes operate in a way which equates language to naming, and so posit a real and knowable exterior world which has been faithfully witnessed.[74] Categories embedded in practices of naming become natural and reified categories of nature itself.[75] Where the testimony involves figures and images, they are invoked so as to bypass the question of whether they should be believed.[76] Instead, the objects of the figure, or even of an entire scientific article, are materialized, and their means of production become effectively black boxed.[77] Linguistic figures which arise in tentative and contingent ways are appropriated and sanctioned

[68]Serjeantson, 1999, p. 200.

[69]Ibid., p. 198.

[70]Ibid., p. 200.

[71]Ibid., pp. 202–203. One can understand the term 'inartificial' as describing something which is not produced, but rather presents itself from the start as a coherent and compelling totality of persuasive reason.

[72]Shapin, 1994, p. xxv. Rotman, 1997, p. 19.

[73]Barad, 1999, p. 7.

[74]Rotman, 1988, p. 25.

[75]Hacking, 1999, p. 161.

[76]Latour, 1987, p. 47.

[77]Ibid., p. 56.

in naturalized rhetoric through a process of catachresis.[78]

The second level of naturalization is that of the perspective of the observer. This god-trick takes the observer out of the observation, and represents her view as unobstructed and unproblematic.[79] At a grammatical level, logicians and scientists substitute context-stripped objective expressions for context-dependent indexical expressions as a means of universalizing the observation.[80] Where language appears to fall short, as it always does (even in the "mythic cartoons of physics and mathematics"[81]), language's own lack of transparency and structural idiosyncrasy can become the excuse. Thus, absurdities such as 5-sided squares can be disbarred, and we can make allowances for partial views on the condition that their shortcomings are merely linguistic ones.[82]

We can begin to disinvisibilize the players in our testimonial drama by reintroducing the audience of testimony in the particular context of science. Scientific witnesses are made to address two audiences: specialists and non-specialists. This distinction is of course somewhat artificial. Such categories as specialist and non-specialist are possible only when we presume the possibility of perfectly commensurable and incommensurable quanta of knowledge by means of an idealized notion of translation. Only then can there be identifiable and stable units of specialist knowledge to which a select group has access, on top of potentially identifiable or stable masses of popular knowledge. In actuality, there are widely varied contexts and means of understanding and sense-making, and it makes more sense to speak of specialist testimony and non-specialist testimony as speech intended for their respective audiences, than to speak of specialists and non-specialists themselves.[83]

[78] Rotman, 1988, p. 17.

[79] Haraway, 1999, p. 176–177.

[80] Lynch, 1992, p. 235.

[81] Haraway, 1999, p. 181.

[82] Hacking, 1999, p. 166.

[83] This is not to argue that specialists and non-specialists cannot be distinguished. Indeed, they are distinguished every day by numerous processes of conventional delineation. Practices of knowledge work and community formation have specialist discourses and positions at their core. The problem comes in distinguishing between specialist knowledges and non-specialist knowledges. Here, discussing specialist testimony in terms of a specialist audience runs the risk of attributing to each audience member a common basis of thought and experience. Such a move fails to consider the multiplicity of discourses and frameworks within a specialty which are united by a sort of Wittgensteinian family resemblance. Moreover, it suggests that certain appeals and rhetorical strategies are capable of

It is possible to speak of all testimony as occurring in some form of creole or pidgin to the extent that all testimony is an attempt to bridge the worldviews of the witness and the audience.[84] Different values and meaning systems require this trading zone model of exchange across lines of ontological difference which can be quite stark.[85] Objects from the experiment are figured as objects of common experience, and so act as boundary objects through which different perspectives can appear to be bridged.[86] Such bridgings are almost invariably interested ones, as translation itself can be seen as a process of onto-conceptual enrollment.[87]

2.5 Inscription and Textual Testimony

Having reintroduced the audience into our picture of witnessing, the next step is to reintroduce the witness. To do this, we must change what it is we think of when we think of a witness. A witness is not merely an observer—someone who sees an experiment and testifies. That narrow interpretation from the observer model misses most of what is interesting about the work of witnessing. Instead, we shall find it profitable to define a witness as anyone or anything which testifies, or, more generally, anyone or anything which is made to testify.

This formulation allows us to work around a number of distracting problems. First, it doesn't force us to separate witnessing from testimony—identifying a witness and identifying testimony are one and the same operation.[88] Second, it remains undecided about the agency of the witness, leaving open the possibility of unwilling witnesses, inadvertent witnesses, and malicious witnesses. Third, it allows many forms of testimony, including written, verbal, and behavioral. Lastly, it leaves an opening for the theorist to enter the witnessing drama as the person who evaluates what testimony is, and,

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resonating with an audience at a purely factual level, which is to risk denying the crucial role of rhetoric and community in producing these resonances.

[84] Galison, 1999, p. 154.

[85] Ibid., pp. 138, 155.

[86] Haraway, 1999, p. 185.

[87] Latour, 1999, p. 259. Latour, 1993, p. 3. Callon, 1999, p. 81.

[88] I should note here that neither the identification of witnesses nor the identification of testimony are, by this virtue, trivial problems. What counts as testimony depends integrally on the social, textual, and political positioning of that would-be testimony, which is quite capable of persisting as a disputed category.

in making testimony, makes things testify.

Since our object here is science, and since the natural and social sciences have a characteristic obsession with graphism,[89] we will use written and graphical productions as our defining model of testimony, and hence of witnessing. A stranger visiting a laboratory could easily conclude that a laboratory is a very expensive factory for publishable papers.[90] Their main means of producing papers is through the creation and combination of *inscriptions*, which take the traces of physical and mechanical phenomena and render them either more mobile or less mutable.[91] These traces, by virtue of their inscribed mobility and immutability, can then be aggregated at centers of calculation, within the lab and within a knowledge system at large, to produce stable and meaningful knowledge.[92]

While the classical model of inscription is the conversion of chemicals and other laboratory raw materials into charts and diagrams by means of laboratory instruments and techniques, it must be remembered that not all charts and diagrams are inscriptions, and not all inscriptions come in diagrammatic, or even printed, form. Rather, inscriptions represent the work of rendering objects (typically laboratory productions) mobile, immutable, presentable, readable, superimposable, and combinable.[93] Paper is particularly powerful as a medium of inscription, for it adds to these properties those of flatness, scalability, textualizability, geometrizability, diagrammability, and enumerability.[94] Among the benefits of inscribed objects is their appearance of being translatable without corruption,[95] a property to which we will return at length in chapter 5.

The appearance of translatability is but one of several material-epistemic transformations brought about by the process of inscription. In a fateful displacement, inscription's promise of uncorrupted reproduction has the effect of shifting questions of accuracy away from the material medium of the text and toward the text's message.[96] A good inscription is one which makes its physically inscribed character appear irrelevant. Inscriptions actively consti-

[89]Latour, Visualisation, p. 15.
[90]Latour and Woolgar, 1986, p. 71.
[91]Latour, Visualisation, p. 10.
[92]Latour, 1987, p. 230. Latour and Woolgar, 1986, p. 51.
[93]Latour, Visualisation, pp. 3, 6. See also Latour, 1987, p. 223.
[94]Latour, Visualisation, pp. 16, 18–20. Latour, 1987, pp. 226–227.
[95]Latour, Visualisation, p. 7.
[96]Ibid., p. 11.

tute our visual world by providing means of defining and perceiving it.[97] It is in this sense that all objects come to depend on inscriptions for their very existence: it is only in the space between inscriptions that an object can be materialized because it is only from within the field of apprehension that the object comes into being.[98] Visual-material culture is thus positioned at the center of the creation of stable knowledge.

These epistemic valuations of inscription, by no means unique to modern science, led Galileo to believe inscriptions in mathematical terms with almost no evident concern for what absurdities such inscriptions might produce. In his ardor for inscription, he made claims on the basis of triangular geometry which attest to physical phenomena which no present day scientist would claim to detect.[99] The outputs of inscription devices and procedures, however, are just the lasting manifestations of a long and murky process where the real work of science is done, endowing a wild and unruly nature with inscription's properties of mobility, immutability, combinability, and so forth.[100] This process is where vast networks of materials and concepts and investments of meaning aggregate around a unit of scientific activity and sediment the scientific objects into inscribed ones fit for publication.[101] Inscriptive work falls into the general ethnomethodological category of rendering practices.[102] Rendering practices create all objects of observation, literally rendering them as objects. This needn't be a passive rendering. Lizards in field studies are routinely strategically amputated to aid with identification, to give one particularly corporeal example.[103]

Observed objects are elaborated according to scientific characteristics,

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[97]Lynch, 1985, p. 59.

[98]Latour and Woolgar, 1986, p. 127.

[99]Latour, Visualisation, p. 23. For more on Galileo's "portentous triangles" (p. 8 of Gillispie), see Gillispie, 1960. In particular, pp. 3–7. In *Il Saggiatore*, Galileo wrote that "The book of nature is written in mathematical characters" (quoted on p. 46 of Gillispie), and we should note that for Galileo, considered by many the founder of modern science and mathematics (including Gillispie, and Edmund Husserl, 1939), nature was constituted *both* textually *and* mathematically. We might call this phenomenon *Galilean witnessing* to distinguish it from Boylean witnessing. Galilean witnessing consists of physically or observationally derived mathematical starting points, which are then manipulated mathematically to produce new physical knowledge.

[100]Pinch, 1985, p. 8.

[101]Lynch, 1985, p. 37.

[102]Ibid., p. 38.

[103]Ibid., p. 41.

which are projected onto the objects in order to become incorporated parts of the objects. Thus, a surveyor will place stakes at intervals in a field in order to construct a geometric grid which will appear in her map as a natural part of the landscape.[104] The process of marking is made to disappear, and a "docile object,"[105] one endowed with a surveyable and comprehensible natural order, emerges in the inscriptions of the scientist as though it had always been. The collection and processing of data functions especially to upgrade the geometricity of the observed objects.[106] Indeed, geometric forms, eminently stable and superimposable, are among the most prized productions of inscription in its capacity as a mathematizer of the material world.[107]

Math, despised by Boylean noblemen for its inflexibility, becomes an ideal of inscription for its mobility and combinability.[108] Where Thomas Hobbes, great champion of mathematics in natural philosophy, used figures at all, they were entirely geometrical.[109] When geodetic surveyors went out into the frozen tundra or the dense tropics in search of the shape of the earth, they always were sure to return with numbers, brought back like hard-won trophies.[110]

2.6 Witnesses in Play

In the previous section, inscription was tied to witnessing through the graphical productions in papers which, quite literally, produce the testimonial narratives of science. Now, we can see in more detail how inscribed objects bear witness in other ways as well. We have already alluded to the ways in which a scientist's testimony brings her audience in to her side of an argument,[111] but the testimony also brings in the objects of testimony, or the elements of the experiment or event in support of the scientist. The speak-

[104] Lynch, 1985, p. 42. Descartes went so far as to theorize perception in terms of a *natural geometry*. Maull, 1980, especially pp. 23–24. Cartesian optics was widely influential in many areas of philosophy, and underwrote many a geometrization and representation. It carried with it its own theory of nature, science, and error. Ibid.

[105] Lynch, 1985, p. 43.

[106] Ibid., pp. 54–55.

[107] Latour, 1987, pp. 243–244.

[108] Porter, 1985, p. ix.

[109] Shapin and Schaffer, 1985, p. 146.

[110] Terrall, 2006, p. 690.

[111] Latour, 1999, p. 259.

ing witness (which must also include the writing witness) sets herself up as a credible interpreter, or spokesperson.[112] The actants, whoever or whatever are represented by the speaking witness,[113] act themselves as non-speaking witnesses.[114] Spokespeople can enlist virtually anything, from microbes[115] to scallops[116] to laboratory technicians,[117] and in so doing they render the actants' work invisible, or rather only sufficiently visible (and not a shade more than is strictly necessary) to be able to support the argument, for if they were truly invisible then an enterprising theorist or historian would be utterly unable to recover them, nor would the spokesperson be able to speak on their behalf.

In a sense, the actor-narrator-witness fabricates the actant[118] in order to gain allies in support of her account, and this fabrication happens by way of inscription. Inscription is powerful. By rendering things mobile, stable, and combinable, one may act on them at a distance.[119] This is because one can, first, transport them from their site of production to other sites of production or contestation, second, be assured that they will remain the same from one site to the next and from one time to the next, and third, be able to incorporate or deploy them in whatever context or role one requires of them. The witness has the power to synoptically gather and dominate a veritable horde of co-witnesses.[120] These immutable and combinable mobiles[121] can come rushing to the aid of a scientist at the turn of a page. And they are a valuable resource: agonistic encounters in scientific production are decided in large part by the sheer quantity of well-aligned and faithful allies each party can enlist.[122]

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[112] Latour, 1999, p. 268. Latour, 1987, p. 71.

[113] The question remains to what extent they are actually represented. See Callon, 1999, p. 76.

[114] Latour, 1987, p. 84.

[115] Latour, 1999, p. 259.

[116] Callon, 1999.

[117] Shapin, 1994, p. xxxi.

[118] This fabrication does not happen from nowhere. Rather, it comes by way of the politics of representation. While actants are generally realized from material things, the process of realization can call into question the extent to which this realization is a true representation of that which it claims to represent. See Callon, 1999, p. 76.

[119] Latour, 1987, p. 223.

[120] Ibid., 226.

[121] Ibid., p. 227. We have already implied that to a certain extent 'immutable' means 'mute,' at least at the stage of the narrativization of testimony.

[122] Latour, Visualisation, p. 5. That more allies should correspond to victory in an

Allies are mustered at each stage of scientific testimony. Particularly in scientific articles, allies guard theoretical points of passage[123] as the scientist-witness fends off invisible opponents with a fortress-like account.[124] After an account is accepted and no longer calls for piece-by-piece interrogation, allies come to look even stronger, as their position-derived strength is tested less and less.[125] At this point, mobilized allies have the effect of sedimenting objectivity itself.[126] Inscription is indispensable in all of this, for it puts into play as many potential witnesses as there are potential objects in the scientist's frame of view.

There are several ideas behind this notion that objectivity is sedimented at the feet of armies of loyal allies. We have, at its most basic, a decidedly social definition of reality which considers it in terms of what a society or bulk of people regard as correct, and where reality is the result of the settlement or non-contestation of disputes.[127] This is not precisely what's going on. A good scientist's allies are, for one, not all people. It is important, and especially central to Latour's argument, that even the non-human allies can be thought of as regarding a particular version of reality as correct.[128] If microbes, or even putative microbes, can be shown to support Pasteur's microbial theory, he is all the better off for it, even if other scientists temporarily disagree. Enrolling and enlisting hypothetical elements of reality go a long way toward sedimenting those same elements as legitimate components of the real world. Seaborg's program of creating new atomic elements is perhaps the purest example of this phenomenon, as an atom comes into existence at precisely the moment (or, according to physical calculations, fractions of a moment before) it is made to speak via the experimental apparatus, testifying on behalf of the experimenter through her amassed data.

Another qualification to this particular formulation of socially-sedimented reality is that scientific consensus is formed, in general, among small groups of scientists, not societies at large. Even within a scientific disciplinary com-

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agonistic encounter is almost a tautology. How do we determine the winner except by counting votes? It is nonetheless an important tautology to recognize, and one which is easily brushed aside by realist science.

[123]Star and Griesemer, 1999, p. 507.

[124]Latour, 1987, p. 46.

[125]Ibid., p. 13.

[126]Latour, Visualisation, p. 18.

[127]Latour and Woolgar, 1986, p. 236.

[128]This non-humanity should not be seen as obviating the juridical nature of testimony. See Latour, 1993, p. 23.

munity, the evaluation of new or competing results is almost always left to a very small and often extremely specialized group of competent reviewers. This takes place in both formal review, such as for peer-reviewed publication, and in the informal allocation of intradisciplinary assent. Such a model of scientific consensus, of course, further assumes that a stable and identifiable sense of assent is the standard by which truth and reality materialize. In contrast, one must allow for the possibility of competing realities, non-settlement of questions, and deficits of consensus in describing the common and socially meaningful forms taken by reality. It is rare either that any particular disciplinarily valid truth is incorporated into the worldviews of most individuals identifiably part of the discipline, or that such truths are utterly without opposition, hold-outs, and skeptics.

Latour defines reality as that which resists,[129] and it is with this notion of resistance that we can muster the right range of interpretations to account for non-scientist witnesses, as well as the intricacies of disciplinary consensus, without letting them dominate the scientists who speak for them and shape them. Knowledge must be understood in the context of cycles of accumulation,[130] among other socio-technical environs. If, as David Bloor claims, most of us are, most of the time, Platonists,[131] then this emphasis on accumulation should leave us with mixed feelings. On the one hand, there is a strong sense within several traditions in the sociology and philosophy of science that the cumulative and accumulative nature of scientific knowledge should indicate a promised *telos*, and that our descriptions should come to resemble natural objects and natural objects should come to resemble our descriptions because reality really retains some ideal externality.[132] In these traditions, accumulation is a sign that there is some reality to nucleate our sedimentations. On the other hand, we leave wide open the possibility that accumulation *is* reducible to accumulation itself, that reality is a wishful philosophical plaything, at best a regulatory ideal, and that since there are no stable social essences there are no such Platonic essences either.[133]

Let us pause for a moment to regard our epistemic position. I have just claimed that reality is an object sedimented by a profusion of witnesses. This account should be familiar. We saw it from witnessing's biggest proponent

[129] Latour, 1987, p. 93.
[130] Ibid., p. 220.
[131] Bloor, 1978, p. 248.
[132] Latour, 2007, p. 139.
[133] Bloor, 1978, p. 257.

himself, Robert Boyle, who saw facts as sufficiently aggregated beliefs underwritten by testimony.[134] Boyle's epistemic program was even more powerful than he could have hoped. It governs not only matters of fact, but the very fabric of reality itself. If my story stops here, then it is ultimately Boyle who comes out to be right. I would not be a good historian or philosopher of science if I let that happen,[135] so I must continue my account.

2.7 Objectivity

I have now restored to visibility our three official players in the observer model of witnessing—the observer, the experiment, and the audience—and proffered my first revision of this model by turning everything therein into one form or another of witness. I aim to bring the theorist into the picture as well, and it should already be clear that the theorist is yet another sort of witness, but before I make this move there is some work to be done. My unveiling began with a visible object and event, and I reintroduced the audience by elaborating the plurality of ways in which testimony about the event engages the essentially heterogenous audience. I then reintroduced the witness by turning the event into a supplementary source of witnesses and then regarding the putative witness as their spokesperson. I must now close the triangle by considering the relationship between the speaking witness and the audience. At the core of this relationship is the problem of objectivity.

Objectivity has come to prominence rather recently as a serious concept for historicization.[136] Porter lists three main types of objectivity with three different standards of evaluation. There is absolute objectivity, whose standard is correspondence the the real world, disciplinary objectivity, whose standard is agreement among a community of professionals, and mechanical objectivity, whose standard is close adherence to a system of observational or methodological rules.[137] Each of these encounters problems, ranging from difficulty in confirmation and evaluation of truth or consensus to difficulty in excluding mediating factors such as tacit knowledge.[138] Daston and Galison add moral objectivity, whose standard is the moral character and restraint

[134] Shapin and Schaffer, 1985, p. 25.
[135] See Latour, 1987.
[136] Daston, 1999, p. 111.
[137] Porter, 1995, pp. 3–4.
[138] Porter, 1995, pp. 3–7. See also, Collins, 1985, p. 56.

of the observer, to Porter's list.[139] This gentlemanly formulation of objectivity found a welcome home in the general moralization of objectivity in the 19th century—so much so, in fact, that it worked its way into the foundations of its companion forms, and formed a particularly central component of mechanical objectivity.[140]

Of the countless criticisms which can be thrown at a testifying witness by her audience, the most damaging might be the charge of theory-ladenness. This was certainly among the most potent elements of Hobbes's arsenal when he refused to credit Boyle's accounts.[141] A theory laden statement is one which fails to adequately divorce the hypothetical unmediated apprehension of nature from the network of interpretations which makes it meaningful. Formulated in this way, theory-ladenness is unavoidable in testimony, and arguably unavoidable in the putative moment of witnessing as well.[142] But common sense and mundane reason are precisely characterized by their incorporation of presuppositions about nature into ordinary interactions, and it is precisely common sense and mundane reason that the good or objective or noble scientist must endeavor to avoid.[143] Thus, it is crucial to the witness's self-justification in front of the audience that she transcend common theory-laden interactions with the world if she is to claim to get at its true nature.

If theory-ladenness is the intrusion of the self into the observation, then for objectivity to combat this condition it must take the form of a technology of distance.[144] In the early modern experimental program, matters of fact were the key epistemic currency, and a theoretically innocent observer was incapable of corrupting matters of fact.[145] Boyle's technologies of witnessing—material, literary, and social—all managed this effect of increasing the apparent gap between Boyle and his experiments, and thus served as objectifying resources.[146] Quantification has since annexed many of the values embodied in Boyle's technologies, serving as the technology of choice

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[139] Daston and Galison, 1992, p. 84.

[140] Ibid., pp. 81, 117.

[141] Shapin and Schaffer, 1985, pp. 87, 112.

[142] Hanson, 1965, p. 25. For Husserl's struggles with this problem with respect to mathematics, see Derrida, 1978, p. 158.

[143] Shapin, 1994, p. xxvi.

[144] Porter, 1999, p. 399.

[145] Shapin and Schaffer, 1985, p. 69.

[146] Ibid., p. 77.

for self-discipline and distance for the better part of the late modern period.[147] A central part of this distance is the removal of agency from the witness, stripping her of both accountability for her knowledge practices and the capability of corrupting her experiment.[148]

As we've described them, all forms of objectivity would seem to have a negative character.[149] They are achieved more by what they leave out than by what they pull in. As a technology of distance, this is appropriate. Where exactly objectivity places the witness is not so important as the result that the witness is far away. As we've indicated, the required distance can be achieved in many ways, and often in ways which would seem at odds with each other. A canonical contrast is between schematic objectivity and mechanical objectivity.

Schematic objectivity distances the witness from the experiment by superimposing the experiment onto utterly independent and abstract forms, such as those from geometry. This move, if successful, makes the observer's position irrelevant, because the experiment is ultimately put in terms of Platonic forms which are by their very essence incorruptible.[150] The problem becomes how to successfully abstract an observation. One standard approach is to identify and quantify elements of an experiment. Quantification gives the witness license to leave things out of the discussion, so safe passage can be made to the realm of ideals.[151] A hefty dose of statistical essentialism clears the way for average images or composite observations to accurately approximate abstract forms.[152] Such representations of paradigmatic figures are particularly valuable in assembling typologies, where extraneous details can and should be removed from images of ideal types. Paradigmatic representation is popular, for instance, in field guides whose aim is to provide a

[147]Porter, 1995, p. ix.

[148]Haraway, 1999, p. 177. Barad, 1999, p. 7.

[149]Daston and Galison, 1992, p. 82.

[150]Latour, Visualisation, p. 24. This essence works in much the same way as that of a matter of fact. Incorruptibility is necessary insofar as no Platonic ideal can do without it, and sufficient in that anything which is sufficiently incorruptible comes to take on the role, form, and function of a Platonic ideal. As with facts, if something is discovered to be corruptible, it must never have been a sufficiently well constituted Platonic ideal. See page 31 for a discussion of Galilean witnessing, which invokes an extreme form of the tactics of schematic objectivity.

[151]Porter, 1999, p. 402.

[152]Daston and Galison, 1992, p. 95.

succinct arsenal of useful criteria for identifying specimens.[153]

Mechanical objectivity, in contrast, distances witnesses from their experiments by reducing the witnesses to automata. This move, if successful, makes the observer fundamentally interchangeable with all other possible observers, thus nullifying the particular effects of the individual's theory-ladenness. In the best case scenario, the witnessing could be performed by a machine, which could be supposed to be ignorant of theory.[154] A purely mechanical witness would not contaminate her experience with models and frameworks, and would not even let judgments of verisimilitude weigh in on evaluations of a trial or event.[155]

The need to judge and schematize doesn't go away, but that responsibility is transfered to the audience by presenting them with multiple reputedly unmediated images.[156] Trust in the representations of the witness needn't be based on blind faith because, like with a machine, the audience always can claim access to the inner workings of the procedure.[157] Moreover, the idealized machine in mechanical objectivity needn't actually be a mechanical machine. Before building his famous mechanical computers, Charles Babbage was an advocate of discrete mechanized tasks which allowed the human computers in his employ to produce objective results.[158] Photographic realism neatly manifests the values of mechanical objectivity, and is, for instance, used in field guides wishing to emphasize the unmediated natural splendor of Nature's flora and fauna.[159] These representations suffer from a great amount of surplus detail, which both bolsters their claims to realism and detracts from their ability to effectively typify what they claim to typify.[160]

The irony of this process is that the whole operation of insulating herself from her theory is performed so that the witness can finally begin to theorize. Her objectivity move protects her data from indict, freeing her to pile interpretations upon it without fearing that her ground will be cut out from under her.[161] Here, the witness encounters a minor dilemma, for the

[153]Lynch and Law, 1999, p. 323.
[154]Daston and Galison, 1992, p. 83.
[155]Ibid., p. 120.
[156]Ibid., p. 107.
[157]Jordan and Lynch, 1992, p. 102.
[158]Daston, 1999, p. 119.
[159]Lynch and Law, 1999, p. 327.
[160]Ibid., p. 333.
[161]Pinch, 1985, p. 13.

more she theorizes, the more she might contribute to the general field of knowledge, but the more she risks her claims being undercut at the level of interpretation.[162] The audience also makes a difference in reporting practices. Specialist audiences who are in a position to provide rival data of their own call for results with fewer interpretations. Non-specialist audiences, who have little basis to challenge anything but the most far-reaching interpretations, are most often given highly interpreted and mediated reports.[163] Success or failure in the distancing project is encoded in linguistic modalities modifying references to different levels of interpretation from audience members and scientific peers.[164]

2.8 Playing the Theorist

Thus far, I have been discussing witnessing primarily on a theoretical and practical level. In order to finally disinvisibilize the theorist, I will need to address it on a social level as well. Unsurprisingly, witnessing is embedded in a range of social practices and conditions. For Boyle, witnessing was constituted in an experimental space, which was simultaneously public and restricted.[165] Experiments could be said to be public at the same time that access to experimental spaces and to the status of witness was carefully regulated and restricted.[166] The public had to be drawn upon to produce public knowledge, especially in light of the juridical analogy by which Boyle produced moral certainty,[167] but it also had to be managed so that control over facticity and certainty remained in Boyle's capable hands.

Epistemological frameworks were thus dictated through a complicated give-and-take between different elements of the public and the social elite. The restoration crackdown on coffee house philosophy is but one literal example of the strict regulation of public intellectual space by epistemic au-

[162]Ibid., p. 23.

[163]Ibid., p. 27. One extreme is in newspaper or other popular explanations of new scientific findings.

[164]Latour, 1987, pp. 22–25.

[165]Shapin and Schaffer, 1985, p. 39.

[166]Shapin and Schaffer, 1985, p. 39. The experimental space wasn't the only so-called public space to which social and epistemological access was highly restricted. The Paris opera house in the eighteenth century is a prime example of the tight and intertwined regulation of both social and physical position. Lambert, p. 46.

[167]Shapin and Schaffer, 1985, p. 56.

thorities.[168] While, in principle, Hobbes could claim that abstract matters such as geometry were free from sects,[169] historical shifts in popular attitudes toward geometry certainly had profound impacts on the nature and status of the discipline.[170] In any event, it can't be denied that any form of reasoning comes highly situated in the particular circumstances of its production, and mathematics is certainly no exception.[171] That Boyle's foremost argument against mathematics is that its esoteric status restricted the potential for mass participation should indicate how tightly domains of access were drawn in his time and how crucial the politics of participation were for anyone trying to make truth claims.[172]

Witnessing goes a long way toward making defined and differentiated experimental and knowledge communities. In Boyle's conflicts with More, it was his privileged community of autonomous witnesses which allowed him to deny More access to truth and credibility.[173] Intellectual division of labor in the form of field delimitation gives a certain competence-derived legitimacy to specialists, enshrining them as privileged witnesses of their proper phenomena.[174] These communities acquire broad freedom to establish their own rhythms, standards, and practices.[175]

Choices of problems and tactics can often be directly traced to discipline-wide preoccupations and interests.[176] Such influences are detectable regardless of whether an individual practitioner works in isolation or in the middle of a large and highly networked group.[177] Testimony takes on a vastly different character depending on whether it is between or within subfields,[178] as we've observed earlier, and this shapes both the manner of testimony and the constitution of the field. Inversely formulated, rhetorical productions from within a community are powerful insofar as they make dissenters feel alone

[168] Ibid., pp. 292–293.

[169] Ibid., pp. 328.

[170] Terrall, 2006, pp. 683–684 describes a Parisian fad for geometry.

[171] Livingston, 2006, p. 41.

[172] Shapin, 1988, p. 23. In fact, Boyle himself was mathematically illiterate, and had a wealth of experience to characterize math as lofty and inaccessible (pp. 26, 41–42).

[173] Shapin and Schaffer, pp. 215, 218.

[174] Bourdieu, 1999, p. 40.

[175] Galison, 1999, p. 143.

[176] Pickering and Stephanides, 1992, p. 146 gives a mathematical example.

[177] Jaffe, 2004, pp. 110–112.

[178] Thurston, 1994, p. 5–6.

and excluded.[179]

Amidst all these social machinations, it is easy to miss that witnessing also entails an economic arrangement. The simple fact is: witnessing is expensive. Information is costly,[180] arguing is costly, and inscriptions are enormously costly.[181] Air-pumps are costly, and the naturalistic engravings needed to depict them in a way that makes them present to a reader are costly as well.[182] In scientific disputes, the cost of disagreeing escalates as in an arms race.[183] Papers build labyrinthine webs of figures and citations which enlist allies of various stripes through the structures of witnessing, and all of these need to be refuted, each at great cost, if the knowledge they undergird is to be contested.[184] Well-established testimonial objectivity has real economic implications when low-level data and their elaborate means of production are called into question.[185] It is not much of a leap (although it over-emphasizes the role of controversies in generating truth) to say that reality itself is precisely the collection of statements too costly to revise.[186]

Even where the cost is not in money or capital, the print (read: inscribed testimonial) tradition of science entails vast and long-term symbolic capitalizations.[187] Whether the currency is honor,[188] eponymy, or disciplinary prestige,[189] scientific labor must be constantly expended in order to participate in a cut throat regime of symbolic capitalism. Witnessing's dissociation of the witness from the objects of her testimony is even amenable to Marxist critiques of commodity fetishism and the alienation of (in this case, semiotic) labor.[190]

Bourdieu speculates that these symbolic economics, and especially the emphasis on professional visibility, might account for the prevalence of visuality and metaphors of perception in the exegesis of knowledge in the social

[179] Latour, 1987, p. 44.
[180] Latour and Woolgar, 1986, p. 238.
[181] Latour, 1987, pp. 69–70.
[182] Shapin and Schaffer, 1985, pp. 32, 61.
[183] Latour, Visualisation, p. 12.
[184] Latour, 1987, p. 48.
[185] Pinch, 1985, p. 26.
[186] Latour and Woolgar, 1986, p. 243.
[187] Latour, Visualisation, p. 11.
[188] Shapin, 1994, p. xxvii.
[189] Bourdieu, 1999, p. 33.
[190] Rotman, 1988, p. 30.

and historical sciences.[191] For critical sociology, its status as a science of witnessing makes assertions which, for instance, declare that 'all reality is in some sense virtual' into the natural outcomes of dominant explanatory resources and regimes.[192] Reality appears virtual because we comprehend reality through the virtual eyes of the witness. Our theories propound the witness because our disciplines propound the witness.

Here, the theorist leaps into our picture of witnessing. The participants in our drama of witnessing and testimony are not just elaborated by the theorist. They are witnessed, enrolled, and taken as allies by the theorist in her own symbolic-capitalist space of production. Official sociology's rhetoric of scientificity, according to Bourdieu, gives it the appearance of rigor and cumulativity.[193] Science studies' embrace of the logic of witnessing, even as it questions it, gives it the appearance of scientificity.

It then becomes imperative to regard the theorist as an inextricable part of any account of witnessing. So situating the theorist demands not that the theorist's perspectives be discredited, but that they be accounted for.[194] Our account becomes recursive: the theorist theorizes witnessing, but a theory of witnessing must theorize the theorist's theorizing of witnessing. This is not, as it might appear, a case of the experimenter's (or theorist's) regress.[195] Rather, it is an invitation to make our account accountable to itself, to inscribe our acts of inscription, and to bear witness to our own condition of witnessing.

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[191] Bourdieu, 1999, p. 35.
[192] Herrnstein Smith and Plotnitsky, 1997, p. 9.
[193] Bourdieu, 1999, p. 45.
[194] Haraway, 1999, pp. 177, 180.
[195] See Collins, 1985, pp. 83–84.

Chapter 3

Cauchy's Mathematics

Having laid out a framework for theorizing witnessing, we shall now set the stage for a discussion of witnessing in mathematics by first presenting a historical example of central importance to the modern development of the discipline. The course in analysis taught by Augustin-Louis Cauchy at the École Royale Polytechnique in early nineteenth century France would fundamentally shape what it would mean to do rigorous mathematics in the nearly two centuries since then. By analyzing the social, historical, and mathematical contexts of Cauchy's work, we can begin to see where witnessing might enter into a general account of how mathematics works.

3.1 Revolution, Reaction, Royalism, and Rigor: A Historical Introduction

Augustin-Louis Cauchy, it has often been observed, began his long mathematical career in the middle of a great tidal wave of political and social changes in the French capital of Paris.[1] He was born at the dawn of the French Revolution, and by the time he enrolled at the age of sixteen[2] with the second highest mathematics entrance exam score at the prestigious École Polytechnique the school had begun a massive overhaul and restructuring to fit within Napoleon's civil service system. He won his teaching post at that

[1]For a detailed discussion of the state of science and mathematics in France during the two decades leading up to the Bourbon restoration and Cauchy's professorship, see Fox, 1974.

[2]Gilain, 1989, p. 5.

same school (during his tenure, the École Royale Polytechnique), a post which he would hold for the bulk of his career, in the early years of French monarchy's restoration,[3] and he was forced to abdicate that same post after refusing to swear an oath to Louis-Philippe after the July Revolution.[4]

While a professor at the École, he was known among the politically active student body as an ardent royalist.[5] His political leanings were not unrelated to his frequent and often heated conflicts with officials and colleagues both inside and outside the École over matters ranging from curriculum content and presentation to professional and professorial practice. It has been said that Cauchy sought absolutes in all facets of his life: his religion (Catholicism), his government (by hereditary kings), and his mathematics (of which I will have much to say).[6] Academically and professionally, Cauchy was surrounded by a profound restructuring of French civil society, bringing with it a partial democratization of advanced scholarship. Higher learning and research, in turn, were themselves being torn between a structural reorientation toward practical matters of social and physical engineering in service of the state and a theoretical reorientation toward disciplinary hygiene and right method. The first few decades of the nineteenth century were tumultuous times for any citizen scientist or professional intellectual.

Less recognized is the great wave of mathematical change, largely precipitated and amplified by these political and social changes, which Cauchy inherited and ultimately helped enunciate. Most mathematicians and historians of mathematics name Cauchy among the founders of mathematical rigor as we know it today. It is true that this iconic figure was instrumental in the shaping and institutionalization of a mathematical practice present-day mathematicians can recognize as their own. But, like all historical truisms, this one is far more complicated than it might seem at first glance. Mathematical rigor most certainly was not Cauchy's idea,[7] nor did it emerge in isolation from the broader context in which Cauchy worked. Cauchy's influence and influences must be placed, rather, within a mathematical world

[3] Periods of restoration seem to go hand in glove with lasting methodological revolutions. See Shapin and Schaffer, 1985.

[4] Belhoste, 2003, p. 92.

[5] Gilain, 1989, p. 22, n. 122.

[6] Belhoste, 1991, p. viii.

[7] Though many have claimed as much, or nearly as much. See Iacobacci, 1965, Bourbaki, 1969, and Bell, 1992 for a sampling of the spectrum of historical and analytic approaches which have led to this single assertion regarding the origins of rigor.

picture that portrays him simultaneously as innovator and reactionary, blind man and prophet.

Even before the Revolution, a civil service[8] meritocracy which drew primarily from the bourgeois classes was laying the foundation for the civil system of education in which Cauchy would thrive. His father, Louis-François, was on the vanguard of this formation.[9] An enthusiastic supporter of whoever happened to be in power, Louis-François caught a lucky break on January 1, 1800, when he was elected Secretary General of the senate which formed after the 18 Brumaire overthrow of the Directory.[10] From this post, Cauchy's father had access to such mathematician-statesmen as Laplace and Lagrange, and he made a special effort to seek their advice on how best to educate his son, who even at a young age showed mathematical promise.[11] Readers, then and now, owe Cauchy's elegant prose at least in part to Lagrange, who insisted that Cauchy not pursue mathematics until he had received a proper literary education, to which the young scholar proved remarkably adept.[12] One must also credit Cauchy's immediate family. Augustin-Louis was a black sheep of sorts: a mathematician in a family of lawyers, and his immersion in a family of juristic reasoners almost certainly had a substantial impact on his later expository and analytic style in mathematics.[13] If the great prophet of mathematical rigor was to be anyone, it would have been Augustin-Louis Cauchy.

As an officially trained mathematician in post-Revolutionary France, Cauchy inherited a very particular mathematical worldview. Perhaps the single most distinctive feature of Western mathematics is its allegiance to what it takes to be its Greek predecessor.[14] The canonized Greek mathematical tradition

[8]The pre-Revolution French academy would often characterize itself as in service of the public, a term with its own changing connotations, although access to the academy was still very much non-public. Terrall, 1999, p. 248.

[9]Belhoste, 1991, p. 4.

[10]Ibid., p. 5.

[11]Ibid., p. 6.

[12]Ibid., p. 7.

[13]Ibid., p. 9.

[14]To this day, it is difficult to find anything claiming to be a comprehensive history of Western mathematics which does not start in Ancient Greece. While it is common to include discussions of predecessors to the ancient Greeks, it is implied or explicitly stated, nearly without exception, that genuine mathematics was the original invention of Classical Europe. Bell, 1992, and Motz and Weaver, 1993 are typical examples of such histories. One can compare this historiography of mathematics to those of Classical civilization

encompasses a range of philosophical and technical matters associated with names such as Thales, Diophantus, Pythagoras, Zeno, and Archimedes. But none stand out quite so much as Euclid, whose *Elements* is among the most widely read and circulated texts in the history of the Western Academy.[15] As late as the 1560s, these ancient mathematicians were counted among a variety of innovators and interpreters throughout Europe, Western Asia, and Northern Africa, but by the turn of the seventeenth century the predominant view across the European mathematical and philosophical community was that pure mathematics, as they knew it and practiced it, was their special inheritance from the Greeks.[16]

This view became more pronounced and dominant during the Early Modern period. Its most devout convert and most passionate priest, up until the tail end of his life, may well have been Thomas Hobbes, for whom Euclid and his *Elements* were of particular iconic importance.[17] For Hobbes and his contemporaries, Euclid came to embody all that was right and good in mathematics and philosophy. By this time, there was a clear and uniform sense of a Euclidean method, which until that time was one of many methodological heuristics at play among practicing mathematicians.[18] According to popular wisdom, Euclid's mathematics began with completely self-evident axioms, postulates, and common notions. From these basic epistemological and methodological foundations, Euclid was said to have erected an utterly unassailable edifice of mathematical truth by systematically deriving

described in Bernal, 1987, and detect similar biases and tactics in each.

[15]Bourbaki, 1969, is typical of mid-twentieth century Whiggish historical assessments of the steady progress of mathematics. Euclid is far and away the most cited pre-modern author to whom modern mathematics is traced by the work.

[16]See Høyrup, 1996.

[17]Jesseph, 1999.

[18]It should be noted that this sense may well have substantially disagreed with how Euclid himself likely saw his own work. We shall distinguish the so-called Euclidean model from the actual work and thought of Euclid, treating primarily the former within the scope of this work. Lakatos, 1979, p. 49, n. 1. Høyrup, 1996, traces the myth of a strictly Euclidean mathematical inheritance to late sixteenth century thinker Petrus Ramus. Mahoney, 1980, discusses seventeenth century algebraic practice and the claims made by scholars at the time for its adherence to the tradition of Greek mathematics. Ramus may have been the first to emphasize algebra as an actual component of Greek mathematics. Ibid., p. 148. For seventeenth century notions of proof in mathematics, see Hacking, 1980. Netz, 1999 and 2002, attempts an analysis which resonates with many aspects of my own, but focusing on the historical and textual record of actual Greek mathematics.

one proposition after another from these basic mathematical building blocks. Such a formulation was of particular importance as a model, exemplar, and method for a controversy-laden natural philosophical field vigorously contesting its own foundations,[19] and has remained a potent paradigm of deductive reasoning in philosophy and science to this day.[20]

As Early Modern controversy faded into Enlightenment triumphalism, mathematical method underwent some marked metamorphoses of its own. Confidence in the steady progress of intellectual inquiry led to the dominance of narrative mathematical exposition.[21] As long as history and knowledge were in lock-step, history became a history of what humans knew and had accomplished, and knowledge became the substance of history's grand march. In Enlightenment France, "mathematical development was a human story of ever-increasing insight."[22] Mathematical histories were allied to natural histories, political histories, and even cognitive histories, as the whole of human development and progress was conceptually inter-mapped and superimposed onto a unified story of collective advancement.

On both a narrative and an epistemological level, references to foundational concepts, and particularly to Euclidean geometry, were despised as stale, pedantic, and pointless.[23] Instead, mathematical works described the genesis of this or that idea, and emphasized how a result could be learned or intuited. A line of mathematical thinking needed only a single thinker to unify it into an acceptable and cogent mathematical narrative. There was certainly no requirement that an exposition begin in the Euclidean style with definitions of central concepts and develop their properties according to the sort of unidirectional logic which would be prized a century later. Mathematics in the eighteenth century found its figurehead in the Swiss mathematician Leonhard Euler, whose corpus of mathematical production runs the gambit of modern mathematical concepts, and whose collected works fill entire bookshelves. Euler was known in Cauchy's time as a reckless algebraic formalist.[24]

[19] Shapin and Schaffer, 1985.

[20] Spinoza was emblematic in adopting a demonstratively Euclidean style for his *Ethics*, a purely philosophical work. See Bourbaki, 1969, p. 23.

[21] Richards, 2006, p. 702. Petrus Ramus anticipated this movement in suggesting that mathematics needed a more humanistic narrative approach (p. 701).

[22] Ibid., p. 706.

[23] Ibid., p. 703.

[24] A 'reckless formalist' may seem an oxymoron at first. I will distinguish between two senses of 'formal', depending on the context. On the one hand, 'formal' refers to mathematical forms. Formal mathematics consists of manipulating mathematical signs and

His wide-ranging results were made possible in large part by his apparently complete lack of concern for the concrete problems and models that made his intuitions meaningful. Instead, he followed his algebraic intuition as far as it would take him, producing in his conclusions heights of both brilliance and absurdity. It is arguable that the extreme degree of creative license evident in Euler's works would only be possible in a narrative system where each step of a presentation was not expected to be policed, justified, and argued.[25]

The French Revolution, for scholars at the end of the eighteenth century, represented a great rupture in the steady progress of history and reason. No longer could they count on a steady accumulation of knowledge-stories to enrich their worldviews. The turn of the nineteenth century saw a rapid decline in narrative mathematics and a corresponding revival of the Euclidean expositional tradition.[26] This revival found its first apotheosis in Cauchy's course in analysis at the École Royale Polytechnique, which was widely lauded for its insistently Euclidean exposition.[27] Cauchy's model of right mathematical practice spread rapidly throughout Paris, though not without opposition, and by the middle of the nineteenth century had become dominant through much of Europe. One goal of this work is to elaborate the conditions both of this model's genesis and of its diffusion.

It need hardly be said that post-revolutionary France[28] was a site of mas-

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symbols according to a symbolic intuition, but without recourse to any representational status those symbols might have. An example of formal mathematics would be using the quadratic formula to solve an equation. Opposing this sense of formal is intuitionistic mathematics, where arguments may frequently be drawn from understood properties of that which a symbol represents. The other sense of formal is presentational. I will almost always formulate this sense with its negation, 'informal'. Informal mathematics consists of heuristic arguments where each step of a process need neither be explicitly identified nor justified. Put this way, Euler was the 'informal formalist' *par excellence.*

[25]It was also important that much of Euler's published work consists of collected notes and ruminations which the mathematician himself did not set out to publish. We can still read much into his seeming disinterest in 'going back and justifying' his results (to borrow from present day mathematical parlance), but mustn't read too much into the particular manner of presentation of Euler's surviving corpus.

[26]Ibid., p. 709.

[27]Gilain, 1989, p. 24.

[28]By post-revolutionary France, I do not mean a France molded after the ideals of the French Revolution, nor do I suppose that such a France inherited any particular set of ideologies or preoccupations. Rather, I refer to a France in the midst of grappling with the material, political, social, and philosophical aftermath of the violent turns of events running from the dawn of the Revolution to the restoration of the Bourbon monarchy.

sive upheaval, a staging ground for epic epistemic battles between revolutionaries and reactionaries. Moreover, it need hardly surprise us that Cauchy's course in analysis contains both progressive and regressive elements. Narrative mathematics gave way to foundational mathematics largely because the mathematical community, deprived of the naturalistic historical basis which had served to ground its arguments in the preceding century, was in need of a new (or renewed) foundation. The proximate cause of this foundational drive was the narrative breach of the French Revolution,[29] after which it was no longer sufficient to appeal to a common march of reason to justify new knowledge claims, but, again, there is more at play than can be accounted for in such a unitary hypothesis.

The hallmark of Napoleon's reign was the systematic bureaucratization of French governance, continuing a process which had its beginnings immediately before the revolution.[30] Much has been made of the rapid rise in record keeping, social arithmetic, and rigid systematization implemented by the Emperor who would master the world by mastering its every detail.[31] This bureaucratization, of course, required a new bureaucratic class—one which was educated through a meticulously reworked system of technical colleges in and around Paris. In the wake of all the revolutionary talk of the liberty of the abstract man to think as he may was the formation of a new wave of men actually so empowered by an expanded educational infrastructure built to staff the rapidly extending arms of state control. Long outlasting the short-lived Empire which put it in place, this educational infrastructure forced open the doors of what had before been exclusively a gentleman's pursuit, and forced it to become, in a very literal sense, the discipline of mathematics.[32] Through the formation and regulation of such educational institutions, the French Revolution and the First Empire had the effect of both professionalizing mathematics, and standardizing its curriculum.[33]

Within this new educational system, Euclidean mathematics took on a dual role. First, it offered a means by which the mathematical establishment and leading establishment figures such as Cauchy could both anchor them-

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This France was populated by a highly heterogeneous array of positions, ranging from liberal Enlightenment optimism, to Revolutionary populism, to conservative monarchism.

[29]Richards, 2006.

[30]See Belhoste, 1991, p. 4.

[31]Hacking, 1982, and Foucault, 1977.

[32]See Alexander, 2006.

[33]Belhoste, 1991, p. 213.

selves as producers of right knowledge. As the method of choice, it offered a foundation for producing new knowledge claims which could supersede the chaos of post-Revolutionary history. This role for Euclidean mathematics manifested itself most immediately within the confines of the mathematical disciplinary establishment. Adhering to a fixed method, and especially a method as demanding and constraining as Euclid's, had a second effect, as well—one more outwardly oriented. Euclidean mathematics helped the mathematical establishment to protect their field from an onslaught of new voices and participants.[34] Mathematical rigor has an inward-looking role of shoring up the process of knowledge creation,[35] but this shoring-up cannot be divorced from its necessary effect of limiting and regulating access both to the tangible institutional grounds of the discipline and to the epistemological grounds on which knowledge claims can be made.[36] Right method is more than merely a restriction of cognitive possibilities. It is also a restriction of participatory possibilities in general.

We can begin to see the outlines of the sort of causal historical and sociological hypothesis called for in Bloor's strong program in the sociology of scientific knowledge.[37] Times of intellectual optimism[38] seem broadly to correspond to informal or heuristic methods, while times of epistemic controversy seem to coincide with renewed attempts at foundational affirmation, which generally take the form of methodological prescriptions and proscriptions. It is possible to follow this hypothesis past Cauchy to the tumultuous birth pangs of the modern nation state which produced the German foundationalists Weierstrass, Dedekind, and Cantor, the Italian logician Peano, and the British pre-war prophets of positivism Russell and Whitehead, as well as before Cauchy to the early moderns such as Hobbes and Boyle, and to Euler's politically isolated corner of Switzerland. More recent moves toward machine mathematics are among many indicators spawned in the disciplinary flour-

[34]Mathematics led a movement across many disciplines in the first half of the nineteenth century to increasingly assert a right to determine their disciplines' content and standards. Ibid., p. 213.

[35]Porter, 1995 and 1999, discusses how this is also, ultimately, outward-looking when it comes to questions of institutional legitimacy and trust-worthiness.

[36]See page 74 for a discussion of disciplinary boundary-making through tropes of rigor.

[37]Bloor, 1976, pp. 2–5.

[38]Optimism or distress can be viewed on many scales. Porter, 1995, connects micro-political and macro-political pressures which reinforced the quantification ideal, arguing that social pressures and disciplinary pressures have more in common than the regimented system of disciplines would at first suggest.

ishing fomented by the Cold War of a willingness to reconsider right mathematics without giving too much pause to the many potential accompanying foundational dilemmas.[39]

There is another historical thesis in this story which requires a more specifically mathematical motivation, but which I shall nonetheless sketch here. It concerns not the philosophical foundations of mathematics, but the practiced methodological foundations of the field. When Euclid wrote about numbers and quantities, particularly in Book VII of the *Elements*, he appears to have had in mind concrete physical magnitudes. At least, this is the mindset from which his Early Modern interpreters seem to have practiced their version of Euclidean mathematics. The constructible universe was, to varying degrees of explicitness, a concrete and tangible world of positive expanses, and mathematics entailed the explicit comparison of two or more such expanses according to a physically intuited system of mechanical manipulations. This perspective led to a number-concept which present day mathematicians would find distressingly inadequate to rigorous mathematics. For example, Descartes unproblematically proposed that every number in the continuum corresponded to a distance and every distance corresponded to a number, without needing to further specify the meaning of either concept.[40] By comparison, present day introductory analysis courses of the sort Cauchy taught, and those at a graduate level, respectively take numbers and distances as foundational axiomatic concepts which are built up from intricate formal systems.

From the geometric intuition identified with Euclid, there arose a new and powerful mathematical formalism, one which abstracted the tangible methods of the Early Moderns to a sufficient degree so as to render them vastly more flexible and combinable.[41] Sufficiently abstracted geometric derivations and sufficiently unguarded uses of geometric concepts in formal and algebraic derivations made way for a formalism which would attempt to leave geometry altogether by the wayside. This new formalism reached a mythic apex in Euler's mathematics, and particularly in his manipulations of infinite sums and products. By following an intuition rooted in simple finite geometric cases, his often baffling infinite derivations tested the conceptual limits of

[39]MacKenzie, 2001, explains the ways in which the foundational considerations of rigorous proof were reconfigured and reapplied in the wake of the digital computer. See also MacKenzie, 1999 and 2004, and Rotman, 2003.

[40]Iacobacci, 1965, p. 84.

[41]See Latour, 1987.

a free-rolling field overflowing with confidence in its own possibilities. The mid-eighteenth century might be considered the first time when algebra was the methodological basis of mathematics in its own right.

Cauchy was quite explicit in his rejection of Euler's methods, but his reintroduction of the Euclidean model came with a special twist. Realizing, no doubt, that formal algebra remained a powerful mathematical method, he appears to have worked to gut algebra so as to give it a new foundation, rather than to advocate for its abandonment. The first textbook published from his course in analysis repeatedly emphasizes, both explicitly and implicitly, the foundational preeminence of geometry. At the same time, the text is substantially algebraic in presentation.

For Cauchy, as for mathematicians preceding and following him for centuries in either direction, geometry had a dual meaning. It was, first, a methodological orientation, an incitement to be Euclidean in the most idealistic sense of the term. A geometer was one who made his starting points clear and his arguments firm and rational. But geometry was also a literal investment in tangible space, especially before the middle of the nineteenth century, when non-Euclidean geometries first gained widespread acceptance. Cauchy's emphasis on geometry meant a commitment to a mathematics based on positive lengths, areas, and volumes. His exposition, though algebraic in presentation, was firmly geometric in justification.

Within his course in analysis, Cauchy undertook the task of developing algebraic rules,[42] particularly regarding infinite sums, which would allow a mathematician to rest assured that his algebraic manipulations were geometrically justified, in both senses of geometry. His work represented a double movement—a geometric justification of algebra coupled with an algebraic re-presentation of geometry—which profoundly affected the conceptual possibilities for both algebra and geometry. It was this movement which admitted the possibility that formal algebraic rules (in Cauchy's case, those derived from geometry) could serve as the basis for a rigorous mathematical practice which was practically and methodologically rooted not in geometry but in algebra.[43]

[42]It has been argued that, rather than algebraic, Cauchy's analysis was primarily arithmetic (Gilain, 1989, p. 24.), although this conclusion is probably heavily informed by models of algebra developed after Cauchy's time.

[43]This basis supplanted previous brands of formalism and empiricism which had been part of various foundational claims in the French academic establishment in the Revolutionary period. Belhoste, 2003, p. 253. Belhoste describes a wide range of influences from

This possibility was taken up by the German foundationalists of the second half of the nineteenth century. Where Cauchy's intuition and justification was resolutely geometric, they turned the tables by insisting that geometric intuition be justified by a formal algebraic foundation. Where Cauchy "sought to give them all the rigor one requires in geometry, in such a way as to never resort to reasons drawn from the generality of algebra,"[44] Dedekind could "resolve to ... find a purely arithmetic ... foundation" and criticize his predecessors for "appeal[ing] to geometric notions or those suggested by geometry."[45] This algebrization of rigor was, in turn, refined into a set-theoretic and formal logic-based foundation which carried mathematics into the twentieth century.

I thus propose that Cauchy's mathematics occupied a crucial junction point in the formation of mathematical rigor and right method as we know it today, but not in the way most historians of mathematics suggest. The typical story is that it was Cauchy who realized the need for rigorous and step-by-step justification of mathematical results which appeal only to what has been already established. Cauchy, the story goes, first saw the need for defining the terms of a proof and sticking to them in a logical and rational manner until the result, precisely formulated in these same terms, was established. The Cauchy of the typical story would not be particularly interesting, for he was neither the first nor the last nor the best to make such claims to rigor and deliberate method.

The Cauchy I describe in this work is truly monumental in insisting on the possibility of giving foundational status to formal operations grounded in right geometric practice.[46] It was Cauchy who took the most eminently available standard of rigor, Euclidean geometry, and the most popular and successful means of knowledge production, formal algebra, and wed the two in such a way as to drastically broaden the scope of geometry and instill trust and virtue in algebraic mathematics. I propose that Cauchy was among the greatest of the great epistemic *bricoleurs*. He took the dominant mathematical-philosophical problem of his time—how to produce good, right, and true results, and how to produce them in quantity—and, through

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within Cauchy's institution on his methods (pp. 252–259).

[44]Cauchy, 1821, p. ii

[45]Dedekind, 1948, pp. 1–2.

[46]This is not to imply that Cauchy did all of this willfully and deliberately. It is eminently unlikely that he could ever have predicted the direction his foundational influences would lead European mathematics over the course of the next century.

a willful combination of two utterly opposed schools of practice, synthesized a mathematics which could strengthen both schools, and ultimately allow mathematics as a discipline to expand in countless directions while retaining a sense of its essential unity and epistemic origins.

3.2 The Mathematics of the *Cours d'Analyse*

We shall now attempt to outline some dominant features of Cauchy's mathematical paradigm, taken in the Kuhnian sense of a set of model problems and solutions which delineate normal disciplinary practice and form the basis for commonly accepted intuitions.[47] When Cauchy was learning mathematics, the field's model problems closely resembled, in many ways, those attributed to the ancient Greeks. Mathematicians were still hard at work searching for roots of polynomial equations of low degree,[48] constructing many-sided polygons, squaring circles and doubling cubes, and testing what could be derived from the comparison of ratios and sizes. Several new problems had also come into prominence, and foremost among them was the description of dynamic phenomena such as the propagation of waves on a string. Cauchy's early work drew from a wide selection of these problems, and by the time he had established himself at the École he had published on, among other topics, symmetric functions (used in the study of roots of polynomial equations), polyhedra, and waves. For Cauchy and his contemporaries, the practical geometric intuition which informed their problem selection and solutions largely came from classical case studies, such as the construction of inscribed polygons.

[47]Our particular focus will be Cauchy's mathematics as it appears in the first years of his *Cours d'Analyse*. It would be a mistake to imply that Cauchy's mathematics was uniform or stable during his career at the École Royale Polytechnique. See Gilain, 1989, p. 22. The period on which we shall focus has been resistant to detailed historiography, due in large part to the paucity of surviving student notes from the first years of Cauchy's course. Ibid., p. 28, n. 162. Since his time, Cauchy's work has continued to undergo reevaluation and reconsideration as it has been incorporated into theorems and histories. For one example from within the École Royale Polytechnique shortly after Cauchy's death, see Gispert, 1994, p. 183. Belhoste, 2003, p. 231 attests to the broader significance of Cauchy's lessons within nineteenth century mathematics, despite their difficulty and inaccessibility for most of his students.

[48]Belhoste notes that Cauchy's 1815 work on the calculus of substitutions is rare in ignoring this dominant problem area. Belhoste, 1991, p. 32.

Augustin-Louis Cauchy received his own instruction in higher mathematics from the same institution at which he would make his most enduring contribution to the field. While he was at the École Polytechnique, beginning at the age of sixteen in 1805 and departing in 1807,[49] his mathematics professor was Lacroix, in many ways his institutional predecessor, and his *répétiteur* was Ampère, who would be a frequent collaborator and ally of his after he joined the faculty.[50] The École of 1806 was just beginning a massive reorganization and militarization under the watch of Napoleon.[51] By this time, Lacroix had firmly established a curriculum in analysis based on a geometrically derived concept of the limit of a sequence of quantities,[52] a move heavily influenced by the mathematics of Legendre, Laplace, and Bossut,[53] and one which came in tow with a whole set of changes to the practice of analysis, including new problems, methods, and model results.[54] Lacroix had developed his own textbook, the *Traité élémentaire de calcul différentiel et de calcul intégral,* which was already in its second edition by 1805.[55]

The École saw itself primarily as a training ground for engineers, and its academic emphasis changed frequently to reflect prevailing attitudes regarding how best to teach the subject of engineering.[56] Analysis, from Cauchy's time as a student well into his tenure as a professor, was variously seen as anything from a convenient tool for engineering applications to a foundational centerpiece containing the core concepts for any well-founded engineering curriculum.[57] It is significant that Cauchy entered the École as a student just as analysis was entering its apex as a cornerstone of the curriculum, as judged by the amount of formal lesson time devoted to the subject,[58] as he was certainly in the latter philosophical camp.[59]

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[49] Gilain, 1989, p. 5.

[50] Ibid., p. 5.

[51] Ibid., p. 5.

[52] Ibid., p. 5.

[53] Ibid., p. 5, n. 19.

[54] Gilain, 1989, p. 5. In this sense, the analysis of Lacroix's course was a genuine Kuhnian paradigm. See Kuhn, 1996, p. 175. Lacroix's analysis synthesized a vein of dominant intuitions, problems, model solutions, and so forth, to create a coherent framework for doing mathematics within the École.

[55] Gilain, 1989, p. 5.

[56] Ibid., p. 6.

[57] Belhoste, 1991, p. 63.

[58] Gilain, 1989, p. 5. Gillispie, 1994, pp. 34–35.

[59] Belhoste, 1991, p. 63.

Analysis's brief ascendancy would not last long, however. In 1810, Metz, an artillery engineer, was charged with evaluating the École's engineering curriculum, and heavily criticized it for being too theoretical.[60] Among the suggested changes was the replacement of the limit concept with the study of infinitely large and infinitely small quantities,[61] which were considered to be of more practical use in real world engineering problems.[62] Approximate methods more suitable to the particular calculations required of future engineers[63] supplanted staples of the Lacroix program such as the calculus of variations.[64]

Starting in 1810, analysis was gradually de-emphasized by the École's Conseil de Perfectionnement and Conseil d'Instruction. It would not, however, be until the 1820–1821 school year (the year immediately prior to the publication of Cauchy's text) that algebraic analysis saw its sharpest decline as an autonomous part of the course, when the material was dispersed, over Cauchy's objections, to either prerequisites for admission to the École or other parts of the integral and differential calculus curriculum.[65] While algebraic analysis was de-emphasized after its apex in the early nineteenth century, Belhoste observes that Cauchy's course became an extreme example of a general analytical reorientation of the École during Cauchy's time as a professor of mathematics there.[66] This trend was not entirely driven by the École's faculty. Polytechnicians of all stripes rallied around analysis as a challenging and rigorous method suitable for an elite and highly selective school.[67]

The École faced a brief scare on the thirteenth of April, 1815, when the student body was officially disbanded by a royal ordinance after an out-

[60]Gilain, 1989, p. 6.

[61]Through the remainder of this work, I will use 'infinitely small quantity' or 'infinitely small' and 'infinitesimal' interchangeably, following the convention of Laugwitz, 1987, p. 260. They translate Cauchy's phrases *une quantité infiniment petite* and *un infiniment petit*.

[62]Gilain, 1989, p. 7. We should note that Lacroix's textbook continued to be used, despite its basis in the limit concept. Ibid., p. 7, n. 34.

[63]Ibid., p. 7.

[64]Gilain, 1989, p. 7, n. 33. Lacroix expressed sympathy with this new orientation in a September, 1812 statement to the *Conseil d'instruction* in which he laments the rarity of applications of mathematics to life. Gillispie, 1994, p. 39.

[65]Gilain, 1989, pp. 22–23.

[66]Belhoste, 1994, pp. 23–24.

[67]Gillispie, 1994, p. 40.

burst of demonstrations the day before.[68] Placed at the helm of the analysis course after an 1816 reorganization initiated by Laplace[69] and ratified by a new royal ordinance,[70] Cauchy injected his own changes to the curriculum in 1816 and 1817, splitting analysis and mechanics, and devoting the entire first year of mathematics instruction to the former.[71] Ampère was given the second year course,[72] and the two would each follow the same set of students through the two year program in analysis, alternating between first and second year curricula.[73] Coriolis was selected as *répétiteur* for Cauchy's course,[74], though this selection was arguably as much for his political (royalist) and religious (Catholic) affinity with Cauchy as for any particular mathematical brilliance.[75] The program for Cauchy's 1816 course included new discussions on the continuity and discontinuity of functions and rules for convergence of series, and substantially downplayed the use of infinitely small quantities.[76] Many discussions, including those of the convergence of generalized integrals, and an integral criterion for describing certain solutions to differential equations, were completely new to the course.[77] Cauchy's 1817 course notes were dominated by the limit concept, discussed averages and bounds in the limit, introduced continuous functions in the modern sense of the term, and included one of the first known cases where the Intermediate Value Theorem was presented as a precursor to the fundamental theorem of algebra.[78] Other distinctive features in the 1817 course for Cauchy were the definition of definite integrals as finite sums, which were discussed before

[68]Belhoste, 1991, pp. 47–48.

[69]Belhoste, 1994, pp. 22–23. This reorganization made Cauchy and Ampère instructors in analysis, and put probability and geodesy under the direction of Arago. See Belhoste, 1991, pp. 45, 48 for the circumstances of Cauchy's hiring. In particular, he was a convenient choice to replace the known liberal Poinsot.

[70]Chatzis, 1994, p. 99. Chatzis adds that analysis and mechanics were meant to be put under the same professor, and that Cauchy managed to defer mechanics to the second year of instruction.

[71]Gilain, 1989, p. 8.

[72]Ibid., p. 8.

[73]Belhoste, 1991, p. 63.

[74]Gilain, 1989, p. 8.

[75]Belhoste, 1991, p. 61.

[76]Gilain, 1989, p. 8.

[77]Ibid., p. 8.

[78]Ibid., p. 9. See n. 49 for Gauss's influence on the presentation of the fundamental theorem of algebra.

indefinite integrals, and the foundational role given to integral calculus.[79] These 1817 elements were quickly taken up by Ampère in his own presentations, and so became common parts to the mathematics curriculum at the École.[80]

These changes marked the beginning of a protracted struggle between Cauchy, the École's administrative authorities, and Arago, the professor of geometry and applied analysis,[81] over the precise layout and content of the Ecole's mathematics curriculum.[82] Cauchy gave his opponents many grounds for complaint, often irreverently dismissing their curricular prescriptions and extending the course well beyond his alloted number of lectures for the academic year.[83] One cannot ignore the crucial importance of these pedagogical disputes to the overall trajectory of Cauchy's curriculum and the overall form of his rhetoric.[84]

Of particular significance is Cauchy's intense dispute over the inclusion of infinitely small and infinitely large quantities in the official curriculum. Cauchy, following his teacher Lacroix, took limits as the foundational concept for his calculus. If the 1816 course was defiant in suppressing the use of infinitely small and large quantities as a replacement for limits, the 1817 course was positively brash in asserting the centrality of the limit concept throughout the entire course, over and above the strong objections of both

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[79]Ibid., p. 9.

[80]Ibid., p. 10.

[81]It might be added here that Arago was strongly ideologically opposed to Cauchy's royalism, in addition to, though not entirely separately from, his methodological oppositions to Cauchy. Gilain notes the apparent paradox in the comparison between Cauchy's ultra-conservative royalism and Arago's aggressive liberalism, despite their mathematical reputations as, respectively, an aggressive reformer and ultra-conservative holdout. See Gilain, 1989, p. 28, n. 168. Cauchy's relationships with his more liberal colleagues, particularly Prony and Arago, could be strained, but were not nearly so uniformly adversarial as one might expect. Belhoste, 1991, p. 54.

[82]Gilain, 1989, pp. 3, 17, 26. Gilain observes that only recently has it been acknowledged by historians of mathematics that Cauchy's relationship with his institution was anything short of harmonious, nor that Cauchy was anything less than a model figure of French establishment mathematics.

[83]Ibid., p. 11.

[84]Of Cauchy's professional conflicts, the best remembered come from his role not as a teacher but as a gatekeeper for those aspiring to make contributions to the field of mathematical analysis. For a discussion of several of the most famous of these conflicts, see Belhoste, 1991, pp. 55–60. These extra-curricular conflicts seem to have less directly affected the form taken by Cauchy's course in analysis.

examiners, Poisson and de Prony.[85] This was also his approach in the 1818–1819 academic year, which was the edition of the course most closely resembling Cauchy's 1821 textbook.[86] Arago specifically complained to higher-ups in 1818 that Cauchy's students were feeble when it came to differential calculus, and blamed this insufficiency on the peripheral role of infinitesimals in Cauchy's lectures.[87] He and Petit, a professor of physics, demanded that (in their view) useless questions of algebra, such as Cauchy's treasured convergence proofs for series, be excised immediately.[88] In 1818, the École sided with Arago's proposed reorganization of the course in analysis, above Cauchy's vehement objections and ultimate non-compliance.[89] Ampère permanently replaced Cauchy in 1819 on the commission in charge of general programming for the École, indicating the severity of the latter's disfavor among the faculty and administration.[90] Arago tried again during the 1821–1822 school year to change the curriculum to focus on applied analysis and algebra in the first year and topics such as geodesy and social arithmetic in the second,[91] leading in part to Cauchy's vehement complaint in 1823 that his academic freedom, including the freedom to teach analysis as he saw best, was being unjustly jeopardized.[92]

In light of these controversies, the textbook needed to be substantially less defiant in order to pass through the École's administration and reach publication. Thus, limits are presented in the text in terms of infinitely small and large quantities. This presentation is more often than not a thinly veiled translation of the limit concept in terms of such quantities, and the proofs and discussion are largely unchanged from their limit-based formulations. There is evidence that Cauchy's principal discussion of infinitely small and large quantities in the first section of chapter 2 of the textbook was added long after the surrounding sections had been written.[93] It has been argued that

[85]Gilain, 1989, p. 9.

[86]Ibid., p. 13. Most of the textbook's innovations, however, can be traced all the way back to the 1817, or even the 1816 course. Ibid., pp. 3, 9.

[87]Ibid., p. 10.

[88]Ibid., p. 11.

[89]Ibid., p. 11.

[90]Ibid., p. 11.

[91]Ibid., p. 13

[92]Ibid., p. 15.

[93]Gilain, 1989, p. 12. Cauchy has also been suspected of having been required by the École to denounce his program of rigor in front of his students. See Belhoste, 1985, p. 115.

framing limits in these terms obscured a clearly emerging sense of continuity as it is presently formulated,[94] and it is certainly true that Cauchy's rhetorical positioning in his textbook had lasting mathematical consequences derived from the text's reception outside of the École.

To better understand how Cauchy intuited the limit concept, we can look at its first domain of application in the textbook, as well as in the early years of the course, which was the question of how to characterize continuous and discontinuous functions. Just as important and positioned more directly in opposition with his mathematical predecessors is the question of when infinite series are convergent.[95] In the former case, continuity remained an almost entirely unproblematized concept for mathematicians before Cauchy.[96] Several competing notions derived from such physical and mechanical applications as describing waves on a string or the distribution of heat on a metal rod were used more or less interchangeably.[97] The eighteenth century contained several famous debates over the nature of waves which invoked competing concepts, but for the most part the participants in these debates seem merely to be speaking at cross purposes when it came to the question of continuity.

Cauchy's analytic and geometric models of continuity appear to differ slightly (at least in Lakatos's reading). When appealing to graphical or geometric intuition, Cauchy seems to understand continuous functions as those which can be drawn on a graph without lifting one's pencil.[98] This interpretation is consistent with models used in studying the propagation of waves on strings, and included as continuous such canonical functions as the 'box function' which alternates between values of plus and minus one

[94]Lakatos, 1979, p. 132. See also, Belhoste, 2003, p. 256.

[95]Lakatos discusses a famous theorem about the convergence of infinite series of continuous functions as a case study in his 'proofs and refutations' theory of mathematical development. Lakatos, 1979, p. 128. Given Lakatos's subject matter, it is interesting to note that some of Cauchy's first significant mathematical work, attempted in 1811, was on Euler's formula for certain polyhedra which could be inflated, cut, and stretched into a plane by a procedure described in Lakatos's text. Belhoste, 1991, pp. 25–26. Lakatos also cites Cauchy's work in this area, but does not develop any connection between the two bodies of Cauchy's work used as case studies. MacKenzie, 2001, pp. 108–112, offers a clear and informative reading of Cauchy's work on polyhedra, as described by Lakatos.

[96]Iacobacci, 1965, p. 84.

[97]Lakatos, 1979, p. 129.

[98]For the interaction between graphs and equations in Descartes's geometric program, see Grosholz, 1980, pp. 164–166.

and would not be considered continuous in Cauchy's analytic framework.[99] Cauchy was not the first to introduce formal analytic criteria for the concept of continuity. Among the more popular prior conceptions was that a function was continuous if it could be represented by a single analytic expression (an interpretation popular in late eighteenth century developments of the theory of Taylor and Maclaurin series representations of functions).[100] In section 2 of chapter 2 of *Analyse Algebrique,* Cauchy gives his analytic criterion for continuity: that "an infinitely small increase in the variable always produces an infinitely small increase in the function itself."[101]

Formulating continuity in this way allowed Cauchy to directly address convergence problems concerning and using continuous functions. His first theorem about continuous functions is of this nature, stating (in modern notation) that if a sequence of values $\{x_i\}$ for $i = 1, 2, 3, \ldots$ converges to a single quantity x, then the sequence of values given by $\{f(x_i)\}$ converges to the value of $f(x)$. His fourth theorem in this section, the Intermediate Value Theorem, shows just how much he continues to employ a *bricolage* of intuitions about continuity, despite having fixed an analytic definition. His proof of the Intermediate Value Theorem was one of the first such proofs to include an explicit formulation of continuity. The theorem states that if a continuous function assumes two values over an interval then it assumes on that same interval all the values between those two as well. The proof he includes in the text (there is one of a more analytic flavor in the third appendix, which was likely added in 1820 as Cauchy waited for publication to begin) invokes an image of the graph of the function crossing the horizontal line corresponding to a value which the theorem claims the function assumes.[102]

As for the convergence of series, Cauchy stood in the path of virtually all of his eighteenth century predecessors, including especially Euler and several of Cauchy's immediate forebears in the French academy, such as Laplace and Poisson.[103] Where the continuity problem was one of making formal an

[99] Lakatos, 1979, p. 130. When drawn to look like the corresponding shape on a string, the horizontal segments of the box function are often connected by vertical lines. When drawn as a representation of an analytic entity, the vertical lines are generally omitted.

[100] Belhoste, 1991, p. 79.

[101] Cauchy, 1821, p. 35.

[102] See section 6.3 for a detailed discussion of these two proofs.

[103] Belhoste, 1991, p. 51. Cauchy was not, however, the first to criticize Euler's use of non-convergent series. That honor may well belong to Lagrange, from as far back as 1797. See Gilain, 1989, p. 24, n. 130.

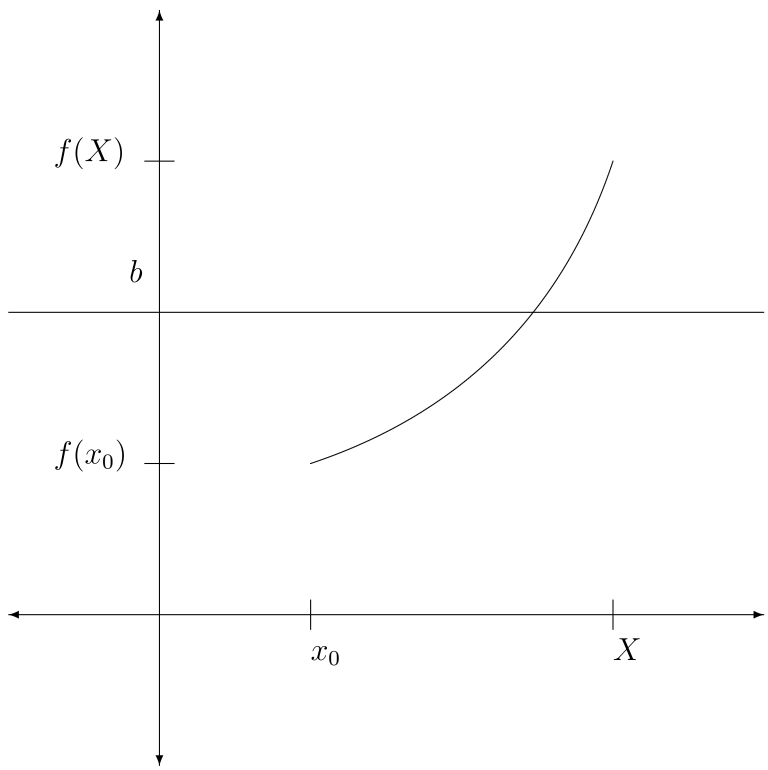

Figure 3.1: The geometric image described by Cauchy in his proof of the Intermediate Value Theorem.

unstated system of intuitions, the convergence problem required reclaiming a geometric intuition for an overextended system of formalities.

The state of infinite series in Cauchy's time must have seemed very dismal to the rigorous crusader. Infinite series were generally treated as formal sums of indefinite extension which could be added, multiplied, and so forth according to the normal rules of arithmetic. This, too, invoked a certain kind of intuition: arithmetic rather than geometric. In a certain sense, arithmetic intuition is more natural to formal symbolic systems, which developed, after all, as a way of generalizing and representing common situations in arithmetic computations. Indeed, arithmetic intuition would later form the foundation of the work of Cauchy's late nineteenth century successors as they sought to underpin Cauchy's geometric intuition with a purely formal system of expressions and relations.

But for Cauchy, arithmetic intuition had no place in the rigorous work of justifying mathematical results.[104] As we observed earlier, Cauchy prided himself in his Euclidean manner of presentation, and for him, Euclidean geometry was the foundation of meaningful mathematics. Euclid's geometry could attain this status because it was rooted in concretely realizable quantities. Common to Cauchy's[105] and Euclid's[106] characterization of numbers is the foundational status of positive quantities. From such quantities, Cauchy goes on to define real numbers and then imaginary numbers, in each case insisting that these are nothing more than formal expressions containing the sacralized positive quantities.[107] These quantities offered the crucial connection between the world of numbers and the world of plane geometry by connecting the basic concept of amount to the basic concept of distance. The deliberate staging of numbers in terms of positive quantities is Cauchy's way of insisting not only on the basic conceptual currency of such quantities, but also on the fundamental position of plane geometry as the source of mathematical rigor.[108]

In terms of series, this meant that a series only had mathematical meaning if it was the algebraic representation of a realizable accumulation of positive quantities. Anything else would be unfounded and unintelligible in a world where the geometer's vision would reign supreme.[109] In algebraic terms, this meant that series would have to demonstrably converge to a stable and fixed quantity. With this in mind, Cauchy gave several formal criteria for the convergence of series. The famous 'Cauchy criterion', that the tail-end of a series become arbitrarily small, is a direct algebrization of the geometry-derived concept that a sum of quantities will stabilize if the total length of

[104]Like Lagrange before him, Cauchy used applications from geometry and mechanics as a conceptual starting point. Belhoste, 1994, p. 23.

[105]Cauchy, 1821.

[106]*Elements*, book VII.

[107]The interpretation of complex or imaginary numbers as existing in a complex plane came after Cauchy, who certainly did not see such expressions as representing anything in themselves, other than the already mentioned formal relationships. See Bourbaki, 1969, p. 203, and Pickering and Stephanides, 1992, pp. 144–147.

[108]Such an insistence on convergence as a pre-requisite for a proper treatment of series had substantial pedagogical effects. Taylor series, for instance, which had been treated early on because of their usefulness in approximation problems, were pushed to the end of the first year course in 1817 in order to allow a sufficient theoretical background on convergence to be developed. Belhoste, 1991, p. 64.

[109]Here, we should recognize the connection between Cauchy's implicit model of a seeing geometer and the observation-centered theory of witnessing discussed earlier. We will return to the idea of mathematical or geometric vision and its connection to witnessing.

the remainder of the pieces to be added tends toward zero.[110]

It is curious to note that Cauchy's geometric treatment of such objects as infinite sums is precisely something that the resolutely finitist Euclidean program would eschew. In fact, the entire Euclidean-mathematics-derived problem of geometric constructibility of certain ratios or the roots of equations was firmly committed to the criterion of solution by a finite sequence of steps.[111] Cauchy's innovation was to allow for the possibility that, under a specially regimented set of enabling conditions, well-disciplined infinite mathematics could be made to behave as though it fit in the much more intuitable finite geometric realm. This double move both expanded what could be considered geometrically meaningful mathematics, by admitting infinite constructions, and contracted what could be considered legitimate mathematics, by excluding from Euclidean justification the wide body of prior infinitist work which did not attend to maintaining a proper finite view policed by the criterion of convergence.

Even more fundamentally underpinning the convergence question is one of what a real or complex algebraic variable may justly represent. I have alluded to what appears to be Cauchy's answer: such variables, and formal combinations between them, may represent nothing more and nothing less than systems of formal relations between positive magnitudes or sizes. Cauchy invests his mathematics at its most basic level in his intuition of spatial distance.[112] One might speculate that this comes from either or both of two distinct and often competing influences. First, it can be said to be a necessary consequence of Cauchy's Euclidean re-centering of mathematics, including the elevation of plane geometry to a primary importance. Second, the centrality of a tangible spatiality in Cauchy's mathematics should remind us that he was trained as an engineer, and taught in a school for engineers, and that problems from the applied sciences were always at least implicitly at the core of his discipline at the turn of the nineteenth century.[113]

This suggests an alternative reading of Cauchy's rigor. While it was certainly true that he fancied himself as a proper Euclidean mathematician in as many senses of the designation as he could fathom, it is also true that an engi-

[110]For more on the Cauchy criterion, see Bourbaki, 1969, pp. 181, 192–193.

[111]We might again note here that Cauchy's own early work on the calculus of substitutions was peculiar in ignoring that field's central problem of constructing roots of polynomial equations. Belhoste, 1991, p. 32.

[112]Iacobacci, 1965, p. 308.

[113]Belhoste, 1991, ix.

neer's intuition persists throughout his work. Time and time again, concrete spatial models appear to serve as a basis for his reasoning about abstract formal mathematics, which was universally acknowledged as his strongest suit.[114] Little could have offended him more than the predominance of a mathematics which was so defiantly unrealizable as Euler's.[115] Little could have satisfied him more than ensuring the universal relevance of his spatial intuition by insisting through the paradigm of convergence that all formal mathematics retain an allegiance to distances in space.

3.3 Positioning the *Cours d'Analyse*: the *Introduction* from *Analyse Algébrique*

If I take Cauchy's non-technical introduction to his 1821 Course in Analysis as a watershed and a benchmark in his emerging philosophy of mathematical rigor, I will hardly be alone.[116] Even in the immediate wake of its publication, the textbook was widely recognized as a must-read for any mathematician interested in rigorous methods.[117] The textbook's introduction, which lays out Cauchy's primary positions and investments, offers a concise and cogent picture of just what Cauchy meant by mathematical rigor, and, just as important, to what and to whom he opposed his new formulation of a very old concept.

3.3.1 The Place of the Course

Cauchy opens by thanking his elder-statesman colleagues Messrs Laplace and Poisson. This gesture is as strategic as it is disingenuous. It is strategic in recognizing the enduring authority of these mathematicians in the mathematical establishment into which the young Cauchy was just entering. Through

[114] Ibid., p. 215.

[115] Cauchy's methodological hostility to Euler is generally quite apparent. Less apparent are the many ways in which Cauchy draws explicitly on Euler's work and style, and especially on the contents of Euler's 1748 *Introductio*. Gilain, 1989, p. 24.

[116] c.f. Belhoste, 1991, chapter 13. The 1821 text is stands among more than 1,500 pages of published material coming directly from Cauchy's teaching at the École Royale Polytechnique released by Cauchy over a ten year period. Gilain, 1989, p. 24.

[117] See Belhoste, 1991, p. 71 for a ringing endorsement from young contemporary Niels Abel.

his father, Cauchy had met Laplace at a time when the latter was literally a statesman in the senate which formed after the uprising of 18 Brumaire and the former was still a young student of substantial mathematical promise. Laplace was an emblematic character during this period of French history—a respected intellectual in the first degree who maintained a sense of civic pride and idealism. Poisson, in turn, was an early supporter of Cauchy's, and helped to introduce Cauchy at a very early age into the company of Parisian professional mathematicians.[118] At this point, Cauchy was yet to have his lasting falling-out with Poisson, which wouldn't come until 1825,[119] and the senior geometer was until his death in 1840[120] a figure of central influence in French academic and mathematical circles. The royalist Cauchy,[121] who must have sensed the discomfort harbored by many of his liberal-minded colleagues at his unusual appointment upon of the restoration of the monarchy, on top of those professors at the École who were already starting to bristle at his manner of teaching and his disregard for the school's methodological prescriptions, could not dispense with the support, real or symbolic, of such figureheads as Laplace and Poisson.

The citation is disingenuous, however, in that Cauchy's very next paragraph positions his mathematics directly opposite that of Laplace and Poisson, who were both known for obtaining results, for instance, by introducing imaginary quantities into their analyses.[122] Cauchy's statement of method in his second paragraph merits close attention, and we will give it its due momentarily.

Before that, however, there are a few more moves of political positioning in the first paragraph which should not escape our notice. Twice in this paragraph, he emphasizes the potential usefulness of his text for students. At the time of its publication, the opacity of Cauchy's lectures was a major point of contention between the instructor and the councils at the École responsible for evaluating both the curriculum and his performance in presenting it. Judging from students' and colleagues' evaluations of his teaching, it is apparent that Cauchy's course was heavily targeted toward the top students

[118]Ibid., pp. 29, 37.
[119]Ibid., pp. 51–53, 175. The dispute was, unsurprisingly, predominantly pedagogical, although part philosophical as well.
[120]Ibid., p. 177.
[121]"...more royalist than the king." Gillispie, 1994, p. 41.
[122]Belhoste, 1991, pp. 50–51.

in his class.[123] His lectures were intensely theoretical and dense, and required a high level of comfort with systematic logical exegesis. As an attempt to increase the course's accessibility to the entire student body, and to facilitate ongoing evaluation of the course's content, the Conseil d'Instruction ordered Cauchy to prepare written summaries of each of his lectures.[124] These summaries would later be compiled and published,[125] and the energy expended in preparing them is one possible reason why the subsequent promised volumes of the Course in Analysis were never produced.

Usefulness, or the lack thereof, was far and away the most common source of criticism of Cauchy's course. It was a favorite barb of the politically liberal Messr Arago, professor of applied analysis at the École and frequent and vocal critic of Cauchy. Arago, in line with the École's administrative authorities, saw the overarching goal of the École as one of training engineers, and saw the course in analysis as consisting of important mathematical tools for engineering. His emphasis was on applications and specific problems in engineering to which analysis could be applied. Cauchy's text, however, is one distinctly devoid of engineering examples. It is indicative of tenets held by the rival faction of professors, to which Cauchy belonged, who saw analysis as not a mere tool among many, but rather as a foundational subject at the heart of all of engineering.[126]

This philosophy formed the basis of Cauchy's arguments for focusing the entire first year program on the basic concepts and methods of mathematical analysis, before any engineering or mechanics ideas were even introduced. His rhetoric in the first paragraph goes beyond this. Not only could it "be useful to Professors and Students of the Royal Colleges" (by which he particularly meant those specializing in various types of engineering), but it could also be of use "to those who would make a special study of analysis." The same text could serve two functions for two audiences. It could provide the analytic foundations necessary for advanced study in any field of engineering, and it could form the basis of a concerted study of analysis in its own right as a mathematical sub-discipline.

The paragraph responds to Arago in one further way. Part of Arago's criticism of Cauchy's course is that it failed to adhere to explicit instructions from the École's administration to make infinitesimals, rather than limits,

[123]Belhoste, 1991, p. 73. Belhoste, 1985, p. 115.
[124]Belhoste, 1991, p. 74.
[125]*Calcul Infinitésimal*, 1823.
[126]Belhoste, 1991, p. 63.

the conceptual core of the course. For Arago, infinitesimals were a tried and true tool which could be easily applied across the range of practical calculations necessary in engineering applications. Where Cauchy writes that he has "been unable to dispense with making the principal properties of infinitely small quantities known, properties which form the basis for infinitesimal calculus," we can read his remark as one colored by a shade of resentment. He has been unable to dispense with infinitesimals because of explicit mandates from the École to the contrary, against his better judgement. Indeed, comparing the Course as published to what is known of the 1818–19 course which it most closely resembles, the section introducing infinitesimals sticks out like a sore thumb, and was most likely inserted late in the preparation of Cauchy's manuscript.[127]

3.3.2 The Place of Algebra

The introduction's second paragraph begins with a famous statement of method:

> Regarding methods, I have sought to give them all the rigor one requires in geometry, in such a way as to never resort to reasons drawn from the generality of algebra. Reasons of this type, although commonly admitted, above all in the passage from convergent to divergent series, and from real quantities to imaginary expressions, cannot be considered, it seems to me, but as inductions so as to sometimes apprehend the truth, but which agree little with the vaunted exactitude of the mathematical sciences. One must likewise observe that they tend to attribute to algebraic formulae an indefinite extension, whereas, in reality, most of these formulae subsist only under certain conditions, and for certain values of quantities they contain. By determining these conditions and these values, and in fixing in a precise way the meaning of the notations of which I avail myself, I make all uncertainty disappear; and so the different formulae no longer present anything more than relations between real quantities, relations which are always easy to verify by the substitution of numbers for the quantities themselves.

[127] Gilain, 1989, p. 12.

Cauchy would have us contrast the ascetic and fundamental rigor of geometry with the powerful but dangerous generality of algebra.[128] We have already examined the role of geometry as both Cauchy's model of right method and the source of his mathematical intuition. For his algebraic analysis to be rigorous, he would need to find some way of giving algebra the elusive epistemic character of geometry. It is precisely because geometric rigor is so elusive for the algebraist or the formalist that Cauchy must pay particular attention "to never resort to" purely algebraic or formal arguments. There are two parallel implications: geometry is hard, offering genuine veracity at the expense of great conceptual discipline; and algebra is soft, infinitely pliable and permitting an endless array of ultimately meaningless curiosities which have no foundation on which to stand, although they may sometimes allude to what might be achievable by the hard work of geometry.

This paragraph, after the initial sentence, has two sections. The first (quoted above) is a list of grievances against algebra, and the second is a list of proposed remedies. At the head of his list of grievances is the most heinous: that algebra is often right, and is in fact useful, despite being a poor representative of the exact mathematical sciences. Contained in this sentence is a stark evaluation of Laplace and Poisson's work.[129] In passing "from real quantities to imaginary expressions" they may have obtained genuinely interesting and valid mathematical results, but they were not performing genuine mathematics.

The next sentence clarifies and establishes Cauchy's intention of geometrically grounding his algebraic work, by complaining that algebraic expressions effectively efface their domain of applicability. One can find a general solution to the equation $x^2 - c = 0$ without ever acknowledging that the solution (at least at the time Cauchy is writing) will only have geometric meaning when c is not negative. At the beginning of the problem, there is a square of area c whose side length is unknown. But thanks to the cruel trickery of algebra, by the end of the problem one is finding imaginary side lengths for

[128]Such algebraic generality was strongly associated for Cauchy with Euler's work, and it is likely this remark was specifically targeted at least in part at the great Swiss mathematician. See Gilain, 1989, p. 24. Belhoste, 1991, p. 51 makes a similar argument that Laplace and Poisson were Cauchy's intended targets with this statement. Cauchy was such a vocal critic of those whose methods he deemed unrigorous that it seems possible to attribute the target of this statement to any of a large number of contemporaries and recent predecessors.

[129]Belhoste, 1991, p. 51.

impossible squares with negative areas—sheer absurdity! In Cauchy's mathematics, precautions must be taken so that a problem should never forget its origins, actual or putative.[130]

Cauchy goes on to fix the precise location of this shortcoming of algebra. What is lacking is a clear sense of the meaning of algebra's formal notations. Without such an understanding, in particular of which values and what conditions are properly embodied in an algebraic expression, algebra's formalism lacks referential meaning, and hence epistemological certainty for Cauchy. A precise and careful exposition, however, is capable of banishing this uncertainty. Here, certainty is set up as a consequence of right method, but not an inherent property of it. Rather, right method is that which eliminates uncertainty. Right method has an essentially negative character: it is self-abnegation, an act of discipline and deprival, which ultimately creates the conditions for truth. This ascetic quality of Cauchy's method appears prominently in his prescriptions for geometrically meaningful methods—what he describes as "useful restrictions on over-extended assertions."

These disciplinary strictures might be compared to the scientific representational ideal of mechanical objectivity.[131] At the heart of mechanical objectivity is the effacement of the observer, whose theory-laden partial perspective is fundamentally incapable of doing justice to an observation. Cauchy's advocacy can't quite be understood as the effacement of the mathematician. After all, it is the mark of a particularly skilled mathematician that she can give her arguments all the necessary rigor and justification. What is effaced, rather, is the method. By giving rules and restrictions in order to maintain geometric plausibility for algebraic expressions, Cauchy allows the mathematician to cross over the fact that she did not in fact directly use geometry to obtain or justify the result. We see the beginnings here of a larger conflation of mathematician and method.

There is a positive character to this discipline as well, though it is always a deferred one. When formulae are disciplined to present only "relations between real quantities," they can always be verified "by the substitution of numbers for the quantities themselves." This *in principle* verifiability can be recognized as a central conceit in rigorous mathematics, and I will discuss it

[130]One can compare this idea to Derrida's grammatological supplement. According to Derrida, Saussure's chief complaint against the written word is that it forgets its spoken origins, and this allows all manner of epistemic treachery to seep in. See Derrida, 1974, part I.

[131]Daston and Galison, 1992. Porter, 1995.

at length as my argument about witnessing in mathematics unfolds.

To remedy the sins of algebra, Cauchy attaches to algebraic expressions a set of conditions of validity. Outside the bounds of these conditions, algebraic formalism is free to run wild and produce all manner of contradictory results. Within the safe confines of these conditions, however, Cauchy could be convinced that formal algebra could yet retain its allegiance to its geometric parentage. Each statement in this section of the paragraph is a specific act of policing the grounds of a certain type of perilous formalism. Divergent series (a favorite of Euler) cannot be said to have a sum. Imaginary equations can only be said to have meaning as pairs of coupled equations between real quantities. If an equation violates the conditions of its formulation it must either be discarded or furnished with a new set of conditions. In short, algebra must wear geometry's clothing. To unmask its duplicity, Cauchy puts algebra in disguise.

3.3.3 The Place of Mathematics

Having protected formal algebra by sealing off a safe portion within the bounds of geometry, Cauchy next attempts the same protection-by-restriction for mathematics as a discipline. Mathematics led a surge in the early nineteenth century of disciplinary establishments across Europe asserting authority over their disciplines' content and methods.[132] This disciplinary enclosure movement had the effect of strengthening mathematical and scientific institutions by strategically restricting their scope, thereby reducing the potential for outside encroachment from the public sphere or from competing disciplines. Such moves partially account for a broad academic trend of increasing specialization and refinement of disciplines and sub-disciplines.

Cauchy's particular view of mathematics comes in the context of his ardent Catholicism, from which he derived a bipartite model of truth. On the one hand, there were the moral truths of scripture and divine authority, and on the other, there were the scientific truths of the natural or observational sciences and mathematics. Belhoste argues that Cauchy's preface places his mathematics squarely at the opposite end of the Catholic moral order as an exemplar of scientific truth.[133] Mathematics for Cauchy is quite rigorously

[132] Belhoste, 1991, p. 213. Alexander, 2006, discusses this disciplinary turn for mathematics in some detail, connecting it to a broader philosophical context, as well as to movements toward specialization and professionalization. See in particular p. 725, n. 22.

[133] Belhoste, 1991, pp. 216–217.

knowable, but knowing mathematics does not tell one about God or country.

I read Cauchy's partition of the intellectual field with less of a unifying gaze than does Belhoste. Cauchy claims to have sought "to perfect mathematical analysis," but he is "far from pretending that this analysis should have to suffice for all the rational sciences." This statement ties together the double act of perfecting mathematics and shielding it from its rational competitors. Indeed, for Cauchy, the only methods appropriate to the natural sciences, for instance, are those of observation followed by calculation, and he explicitly disclaims as much in his Introduction. Truths in the natural sciences, while they may be mathematically derived, are strictly outside the domain of pure mathematics as Cauchy would have it practiced. Their origin is natural, not geometric, so they have neither the vaunted Platonic stability of geometry, nor the basis for right method contained therein. The mathematics of the natural philosophers must be ad hoc and instrumental, for it deals in always partial, always mediated and mitigated observational truths.

He doesn't stop there. Next, he contrasts mathematical truths with socio-political truths—in his example: seventeenth century kings of France. Cauchy's rhetoric presents mathematical truths and socio-political truths as of two completely separate registers, each with their own brand of certainty. Indeed, the seventeenth century kings of France are not only at least as certain as Maclaurin's theorem, but this certainty is accessible to a much broader public than that of mathematical truth. Mathematical truth is not special because it is general. On the contrary, it is special because it is geometric, and geometry remains a science of the privileged.

This implies a double obligation. First, mathematical truths are to be held close by those trained to adequately comprehend them. Mathematics is for the mathematicians. Second, mathematics should not and cannot be applied to questions of religion, morality, and politics. This second statement stands in direct contrast to theorists of a rational civil order, such as Condorcet.[134] In an 1821 letter to Paolo Ruffini, Cauchy singles out Laplace and his theory of testimonial probability as a target of this statement's prohibition.[135] Mathematicians who over-assert and over-extend the boundaries of right method, be it by resorting to reasons drawn from the generality of algebra or by applying mathematical analysis to Cauchy's treasured and sep-

[134]Gilain, 1989, p. 11, n. 59.
[135]Belhoste, 1991, p. 51, n. 27.

arate moral order of truths, endanger mathematics by distancing it from its geometric foundations.

Moreover, while the rulers of France may be a popular question, the validity of Maclaurin's theorem must not be. Within a specialized discipline, it is downright dangerous to submit one's achieved truths to public scrutiny.[136] The rhetorical technologies of mathematical proof and method must retain a veneer of disciplinary exclusivity so that the range of possible contestants to a new theorem can be controlled and managed.[137] Cauchy's mathematics is not for everyone, nor does it aspire to be. A good royalist, he maintains that the most universal of truths cannot be entrusted to the most universal of arbiters.[138]

Mathematics, to become and remain stable as a discipline amidst other disciplines and other domains of knowledge, must build good fences which both keep mathematics in and keep non-mathematics out.

[136]Porter, 1995 and 1999, discusses this phenomenon in the field of accounting, where there is a taxing double-pull to be both accountable to the mathematics and accountable to the public.

[137]c.f. Hunt, 1991, p. 74. Hunt describes a case study from late nineteenth century British mathematics in which claims to mathematical rigor are explicitly used to exclude a would-be participant from the field. Warwick, 2003, describes a range of institutional and pedagogical transformations at Cambridge during the same time period which had lasting effects on the theory and practice of mathematical physics, including the disciplinary organization of and standards of rigor within the community of mathematical physicists. See Jesseph, 1999, p. 431, for another example, from Hobbes's time, of disciplinary boundaries enforcing a properly reverential social structure. Yet another dimension of this issue is described in Terrall, 1999, p. 247, which points out that by the 1780s the analytic language of mathematics and mathematical physics was a highly specialized one, meaning both that few could understand it and that it was in a special position to access philosophical truth.

[138]Mackenzie, 2001, p. 304, explains the place of trust in a professional body, which relies on trust relations between small groups of personally known colleagues. Terrall, 1999, p. 246 discusses this paradox of universal access to reason despite the highly restricted access that exists for authorizing discoveries, as well as the attempts of the French academy to distinguish itself from popular science, in part through the use of algebraic analysis. Much more shall be said on these disciplinary communities in later chapters.

Chapter 4

Witnessing in Mathematics: A Semiotic Model

Let us now turn to the practice of mathematics and the problem of its rhetorical production and dissemination. This chapter will present the first of three general models we shall consider. Within this first model, that of mathematical semiosis, I shall be most prone to taking mathematicians at their word. The goal will be to trace the systems of explicit and implicit meaning embedded in the outputs of mathematical work. This chapter is about the meaning of mathematics in the terms in which mathematicians understand it. I shall, for the moment, take for granted the difficult distinction between signifier and signified, as this is a more or less central conceit of semiotics, and temporarily bracket the problems that arise therein until a more thorough treatment of the theory of language and social enunciation can be developed in subsequent chapters. This chapter is about the putative meaning of mathematics, about its representational structures and strictures, and about the relationship between mathematical ideas and the mathematical discipline.

4.1 A Divided Practice

In the relatively small corpus of critical works attempting to historicize and denaturalize mathematical knowledge, be they primarily historical, sociological, or philosophical, there appears to be a common method, founded in the lived work of mathematical production.[1] By realizing that even eternal

[1] c.f. Livingston, 1986, p. 14.

truths need enunciation, and by such virtue can be interrogated with respect to their status as eternal truths, theorists and researchers with a wide range of theoretical commitments have been able to chisel at the towering onto-epistemic edifice presupposed by their forebears. We shall first approach the question of how mathematics is enunciated through its enunciations' representational status within the community of provers and producers. Within this community, there is a basis of structuring tropes and assumptions which prefigure any mathematical output. The first assumption we shall consider regards the process by which mathematics is produced.

Traditionally, mathematical production is considered to take place in two phases: *ideation* and *articulation*. Of course, these stages are never strictly separable, and a large number of attempts to theorize mathematics make the mistake of either ignoring one phase or neglecting the ways in which the two phases interpenetrate. Nonetheless, narratives about mathematics seem to take as given that a mathematical idea is first conceived and only subsequently communicated. Embedded in this supposition is the hypothesis that mathematical communication, while it can aid or hinder understanding of mathematics, is ultimately peripheral to the supposed real work of mathematics, which is coming up with new theorems and problem-solving methods.

The phase of ideation involves the creation of a mathematical result or theorem. Depending on one's school in the philosophy of mathematics, this can involve anything from a purely abstract mental apprehension of a mathematical truth to the development of a new sequence of formal sign manipulations on a page. [2] Ideation is, in its pure and ideal form, a completely solitary mathematical activity. It is this ideal which is responsible for the well-worn stereotype of the lone mathematical genius.[3] Mathematical apprehension and comprehension is figured as a purely mental activity,[4] and, keeping with the Cartesian thesis of a separate and internal mind, mathematics is thus figured as purely internal to each single individual mathematician.

Articulation entails the production of mathematical communication, often in written form. If ideation can be thought of as the private production of mathematics, articulation is certainly the space of the public production of mathematics, and is, moreover, where a proof statement's precise form

[2]These extremes correspond, respectively, to Brouwer's intuitionism and Hilbert's formalism. Rotman, 1988, pp. 4–5.

[3]Jaffe, 2004, pp. 100–102. Alexander, 2006, pp. 714–726.

[4]Blind mathematicians are often called up to underscore this point.

emerges.[5] The particular importance of writing to this articulation is debatable. A written proof is accorded almost mythic status as the ultimate test or criterion for mathematical truth.[6] This standard is reflected in the everyday practice of mathematics, where it is common to say that one has figured out an argument, save for writing out the details, or that one only needs now to set a rough argument to paper. This criterion is not the ultimate arbiter of mathematical truth or falsity, however, and it is not hard to find examples where looser standards of sub-disciplinary comfort or familiarity with a result will sanction it as well as any piece of paper.[7]

4.2 Mathematical Semiosis

In any case, it is fair to say that as abstract forms around which to nucleate our analyses of mathematical production, these phases of ideation and articulation offer a viable and easily complexified entry point into the complicated mesh of knowledge relations which permit the materialization of mathematics. With this in mind, we shall begin by describing a model of mathematical semiosis[8] which bears a strong resemblance to the observer model of witnessing. This model is robust in its potential to analyze both phases of mathematical production, due in large part to its emphasis on relations between symbolic, ontological, and epistemic categories of reasoning and representation common, in principle, to any site of mathematical work. Woven into our discussion of this semiotic model, one should recognize some

[5] Lynch, 1992, p. 245. On the distinction between practices of rigor in public and private mathematical production, see Porter, 1995, p. ix. For Hobbes's distinction between private and public reason, see Shapin and Schaffer, 1985, pp. 103–105. The main point is that the forms and standards of knowledge we keep for ourselves are invariably substantially different from the ones we allow to generate matters of public knowledge. This distinction makes any unitary theory of knowledge practices and production fraught with difficulty. Wittgenstein sidesteps this difficulty by placing public articulation in a primary position. Even so-called private communication, for Wittgenstein, is already and irreducibly couched in the terms of public communication, regardless of it's particular venue. Without judging the primacy of either form of communication, this work will insist that there is at the very least a putative distinction between public and private articulation, that this distinction is routinely made in mathematical narratives, and that different valuations of such modes of articulation make meaningful differences to the practice of mathematics.

[6] Jaffe, 2004, p. 106. Mackenzie, 2001, p. 11.

[7] Thurston, 1994, p. 8.

[8] First elaborated in Rotman, 1988, and expanded in detail in Rotman, 1997.

familiar characters from the observer model of witnessing.

Rotman begins his semiotic model of mathematics with a Peircian straw figure. Peirce, who, after all, was a mathematician long before he was a semiotician, would begin by positing an entity called the Mathematician[9] who speaks in inclusive imperatives—expressions such as 'consider the set S' or 'let H be a Hilbert space.' The Mathematician is, grammatically, the figure in the proof who invites the reader to participate in the proof process. It does so by establishing the terms and context of discussion, and delineating the space of production.

In Rotman's Peircian model, the Mathematician addresses another imagined figure called the Agent, who speaks in and responds to exclusive imperatives—expressions such as 'add the numbers from 1 to n' or 'permute the elements of S according to left-action by the group element σ.' The Agent carries out the ideal, physically inaccessible operations of a mathematical argument.[10] The Agent's grammatical position is to *do* the mathematics in the imagined space of mathematical production.[11] Expressions involving the Agent are explicit and algorithmic, and do not invite participation so much as they demand action.

To account for the actual context of articulation, Rotman re-imagines this model to include a third figure, the Person, who posits an abstracted version of itself in the Mathematician, and who speaks the language of meta-mathematics, including the many other grammatical forms found in a typical mathematical exposition.[12] The Person is the speaker of everything which is supposed to be peripheral to the mathematics of the proof. This includes any heuristic justifications for a method, any context within other problems or examples from the discipline, and any common language expressions connecting, qualifying, or speculating about different parts of the proper mathematical argument.[13]

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[9] References to the Peircian Mathematician will be capitalized, to distinguish it from the mathematician of common parlance. The same distinction will apply to other such figures described by Rotman.

[10] Rotman, 1988, pp. 8, 11.

[11] We should distinguish between the sense in which the Agent *does* mathematics and the sense in which a mathematician does mathematics. The latter does mathematics by engaging in a set of practices which, as we shall see, are in almost constant reference to an invisible realm of action by the former.

[12] Ibid., p. 12.

[13] The conventions, methods, and model problems of a discipline play a large role in determining its viable heuristics. See Rotman, 2003.

There are, for Rotman, three levels of mathematical signification to accompany the three semiotic actors of mathematics.[14] The Mathematician speaks in the mathematical Code, which consists of all rigorous statements which are properly mathematical, be they formal combinations of quantifiers and variables or intuition-based arguments in official mathematical language.[15] The elements of the Code allow the Mathematician to articulate a relationship to any brand of mathematical ideality to which it subscribes.[16] The Code sets the mathematical context of the proof by providing a system for describing the concepts at play, and for connecting different mathematical concepts into a coherent conceptual unity. While the Code exists to deal in relations between mathematical concepts, it is unable to articulate the specific relations which allow elements of an ideal mathematical realm or system[17] to be recombined to produce new mathematics—in other words, the relations between the Mathematician and the Agent.[18]

For those, there is the MetaCode, accessed by the Person, which refers to un-rigorous mathematical language and which is considered epiphenomenal to the actual production of mathematics. MetaCode is essential for establishing the intuitions which, for instance, allow a mathematician to translate the Mathematician's tangible instruction-statement into the Agent's impossible imaginary math-work. It is also the language in which the proof-story is told, offering broader context and justification for mathematical work. Lastly (a later addition in Rotman's work on mathematical semiotics, although it is prefigured in his first essay on mathematical semiosis[19]) there is the Virtual Code, accessed by the Agent, which refers to the space of all legitimately imaginable mathematical operations, including especially those which are impossible in practice.[20] This is an important theoretical addition

[14]Rotman, 1988, p. 15. Rotman, 1997, pp. 22–23.

[15]MacKenzie, 2004, p. 75.

[16]Rotman, 1988, pp. 4–5.

[17]We might call these 'intraconceptual elements', for they exist within a mathematical system already set up as coherent and with stable meanings for its constituent objects.

[18]Ibid., p. 16. For instance, there is a great gap between saying 'let H be a separable Hilbert space and $\{f_i\}$ a countable basis for H,' a statement within the Code laying out mathematical concepts involved in the discussion, and giving the instruction 'scale each basis element according to the coefficient e^{-i},' which is a statement by the Mathematician asking the agent to recombine elements of the already-established system.

[19]Ibid., p. 17.

[20]For example, the Code can tell you what a factorial is, how to compare factorials, and what other concepts depend on factorials, but it cannot, in general, compute factorials,

to the previous two Codes, for it acknowledges that the range of acceptable thought experiments which go into mathematical proof far exceeds the range of such experiments which can actually be performed, mentally, physically, or otherwise.[21] The Mathematician-Agent relationship will emerge as a central resource and problematic in mathematical witnessing.

4.3 Virtual Subjectivity

As we did for mathematical production with the ideation-articulation distinction, we can break down mathematical witnessing into two scenes about which to model the testimonial communication which takes place. The first, more familiar scene is that of the Person-Mathematician speaking to its audience. This discussion takes place in a combination of Code (usually, a mathematical proof) and MetaCode (usually, the Peircian 'leading principle' or motivation for the proof).[22] We might call the site of the Mathematician or proof-giver's testimony the social space of mathematics. This space is where mathematics is communicated, and communicatively constituted within a community of mathematicians.

There is an interesting play of subjectivities in this social space. The prover, as witness, adopts what is made to appear as a bipartite structure in her proof.[23] In this structure, the lived work of proving is communicated as by a Person to an audience of People. People are allowed to discuss heuristics, intuitions, and motivations among themselves, provided they don't come so close as to contaminate the proof account. Spoken by the Mathematician to an audience of Mathematicians, the formal and rigorous proof account presents the putative mathematical substance of an argument. If performed

and it certainly could not compute the factorial of any large number.

[21]We shall return later to the importance of thought experiments.

[22]Ibid., pp. 14–15.

[23]Lynch, 1992, p. 224. Spinoza's *Ethics* polices the boundary between these parts in an especially explicit way, separating axioms, propositions, proofs, etc. from scholia, or explanatory notes. Mathematical papers from Cauchy's day to the present police this distinction to varying degrees and in various ways. Different presentational and rhetorical conventions flag the register in which a statement is intended to be received. Behind the partition of mathematical narrative is the fact that not every part of a mathematical proof is important to every reader, and what each reader values in a proof can vary quite widely. This bipartition represents two extremes of priority-making: emphasizing either the strictly logical passages or the strictly heuristic ones.

successfully, the metamathematical discourse[24] amongst the People is made by the Mathematicians' discussion to seem entirely epiphenomenal to the production of genuine mathematics.[25]

Sitting just behind these two intertwined conversations is the second scene of mathematical witnessing. There, dozens of spectral mathematical Agents dance in the minds of the People-as-Mathematicians, performing in each instant countless infinities of mathematical operations of every imaginable stripe, all under the watchful gaze[26] of the Mathematician. We should note from the start several problems and caveats with this image. First, the range of what the Agent is capable of doing varies vastly from one school of mathematical philosophy to another. One ongoing contemporary controversy is, put in these terms, over whether the Agent can simultaneously go through an arbitrarily large collection of sets and select a representative from each one. The Agent is still far more powerful than an ordinary human or Mathematician. Few Mathematicians would deny the Agent the power of simultaneously performing the selection for, say, a number of sets exceeding 10^{100}, which is approximately the number of ways to randomly select any three atoms in the universe. Moreover, the Agent can do what no Mathematician can do, and certainly what no Person can do, in considering synoptically all possible manifestations of a single mathematical concept. Where a Person can verify that $1 + 1 = 2$ for stones, or cats, or mosquito bites, only the Agent can verify that $1 + 1 = 2$ *in general*.[27]

Second, there is a substantial amount of work which needs to be done in the MetaCode to make the Agent a credible simulacrum of a supremely powerful Mathematician.[28] A great many justifications and rationalizations go into every action of the Agent, who can clearly be said to have no autonomous power of its own. For if it did, it could no longer act on behalf of the willful

[24] Metamathematical discourse is that which is peripheral to the rigorous work of mathematics, and thus outside the proof. This is the primary sense in which Rotman uses the term. Sometimes, 'metamathematics' can also refer to work on the logical underpinnings of mathematical arguments. Alas, even mathematics admits autantonyms: terms which can, depending on their context, have opposite meanings.

[25] Rotman, 1988, pp. 14–15.

[26] Here, *under* must carry the connotation of *beneath*, in every sense, and also suggest that by being under the gaze the Agents are in a meaningful sense missed by the gaze, as well.

[27] Ibid., p. 20. We shall see that the sense in which the Agent *verifies* a proposition is a very special one, and a very powerful one, as well.

[28] Ibid., p. 16.

Mathematician, and in any event couldn't be trusted. Closely related is a third problem, that the model overstates just what it is the Agent does, or is witnessed to do by the Mathematician. In the mathematical imaginary, the prover's chartering of such mathematical simulations most often involve the Mathematician imagining itself (as Agent) performing a sufficiently representative example[29] of the desired operation, and then leaving the Agent on her own to work at infinite speed without supervision. The Virtual Code isn't important in mathematical production except insofar as it exists. But its existence is crucially important to mathematical production. It is what gives power to the innumerable 'and so on's which populate any discourse making claims as vast as those of mathematics. One must not forget that the moment of saying 'and so on' is the moment at which the prover and her audience cease to continue onward.[30]

This suggests a fourth problem concerning the semiotic level at which proofs take place. Rotman's illustrations of his semiotic model would lead one to believe that every formal mathematical operation is actively imagined to correspond to a meaningful sequence of more or less concrete manipulations in the mathetic realm of the Agent.[31] This is hardly the case in actuality. Mathematical proofs and the mathematical witnesses who use and certify them trade signs at several levels of semiotic meaning, including those of purely formal manipulation, rote evaluation and substitution, and heuristic or intuitionistic (rather than Agential) imagination. Lastly, we should remember that each member of the audience and producer of a proof need not imagine the same Agent doing the same actions. The work of the Agent is, at least in practice, not set in a logically deterministic manner. Rather, it depends on what each Person construes, from her own experience and in the context of epimathematical arguments from within the proof at hand, to be the relevant exemplars and operations. It is not even a foregone conclusion

[29]We shall revisit the idea of representative examples in our discussion of diagrams, below. For now, suffice it to note that the choice of an example to stand in for the whole range of mathematical objects in question is extraordinarily important in defining the heuristic for a result, delimiting the domain of its applicability, and informing the development of further mathematical work.

[30]Structures such as this, including the *et cetera* clause, are also present in ordinary language, and often do similar work. As Garfinkel argues throughout his works, people do not explore sources of agreement. Of interest in the mathematical example is the way in which the work of the *et cetera* clause is allowed to produce rigorous, true, and meaningful results.

[31]Ibid., p. 12.

that different Agents put to the same virtual task will arrive at the same result, and much of the work, both in the Code and MetaCode, of mathematical argumentation is in creating equivalences between these disparate outcomes.

With these reservations in mind, the scene of the mathematical Agent remains a potent image for exploring the range of subjective agencies invoked in the process of mathesis.[32] In the production of a proof, the mathematical audience is asked to give assent to arguments in multiple realms—heuristic, formal, virtual, and so on. Each realm of argumentation requires for its existence the hypothesis that it is entirely independent of the other lines and spaces of reasoning. At the same time, each realm depends integrally in its production on the others. The Agent and the Person are made, against their will, to speak to each other in mutually unintelligible languages. As a literary technology, the mathematical proof must invoke an extraordinary number of witness relations in order to fabricate mathematical reality. These witness relations are both material and virtual, and they are for the most part embedded in the grammatical and linguistic form of the proof.

4.4 Virtual Witnessing, Virtual Code

We have, of course, seen this phenomenon in a slightly abridged form under the name of virtual witnessing.[33] By consigning physical space to abstract representations thereof, virtual witnessing manages both abstract and physical space.[34] The difference with respect to mathematics is that the 'physical space' of math is itself abstract, even inaccessible. Virtual witnessing in mathematics entails managing the fundamentally inaccessible quasi-physical experience of the Agent *through* the management of the abstract Coded work of the Mathematician. It is in this sense of semiotic abstraction accessing the Virtual Code that mathematical arguments can be said to be built upon virtual writing.[35] Indeed, the Mathematician's (i.e. its Agent's) reality is an entirely virtual one, and depends critically on the semiological homology

[32]I mean this term in a slightly more restricted sense than Foucault, 1970, p. 73. Here, I refer to a broadly understood notion of mathematical order, rather than Foucault's expansive calculative order.

[33]Shapin and Schaffer, 1985, p. 60.

[34]Ibid., p. 336.

[35]Rotman, 1997, p. 37.

between mathematics and virtual reality in general.[36]

The essential virtuality of mathematical reality has important consequences. Foremost among them is the idea that mathematical reality is fundamentally pliable—that it doesn't resist attempts to define and know it in the same way or to nearly the same extent that material reality does.[37] Within this model of pliability there remain constraints. Math has momentum, in a certain sense, defined by the spectrum of forced or constrained moves, regulated by institutional methods and knowledges, found in the production of mathematical ideas.[38] Mathematical production can be described as quasi-experimental: a range of theoretical inputs is tested against the conventions and expectations inherited from right mathematics, and the results are only later constructed in terms of mathematically rigorous arguments.[39]

Perhaps it was this strange mixture of determinacy and pliability, experiment and theorization, that led Boyle to insist on the distinction between actual experiments and thought experiments.[40] Boyle was especially critical of thought-experimenters Hobbes, Spinoza, and Pascal.[41] The very language of mathematics allies it closely with the tradition of reflexive thought experiment,[42] and the quasi-experimental work of the Agent, particularly in the ideation phase of mathematical production, takes the semiotic form of a generalized thought experiment. This is particularly the case at the practiced level before the invocation of the full force of the Virtual Code.

A thought experiment becomes a mathematical thought experiment, however, at precisely the point of where the Virtual Code is invoked. In other words, a thought experiment becomes mathematical when we feel safe enough to leave the Agent to its own devices—when we can entrust our experimental work to the Agent, and let the Agent finish the experiment on our behalf. We can do this because the Agent is not willful. Rather, it is a non-reflexive

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[36]Ibid., p. 35.

[37]Pickering and Stephanides, 1992, p. 142.

[38]Ibid., pp. 140–141.

[39]Glas, 1999, p. 3. Hunt, 1991, p. 81.

[40]Shapin and Schaffer, 1985, p. 55. Mathematics is experimental in a sense beyond containing elements of thought experiment. Experimentality in mathematics alludes to the whole range of experimental and quasi-experimental activities, from forming and testing hypotheses, to drawing conclusions and modifying frameworks, which are shared by mathematics and the natural or experimental sciences.

[41]Shapin, 1988, p. 44.

[42]Rotman, 1988, p. 10.

rule-follower.[43] The Mathematician can leave the Agent alone precisely because the Mathematician can be confident that she knows, if not what the Agent is doing, at least how the Agent is doing it at every instant.

This formulation is not quite right. That the Agent is a rule-follower does not alone make it a reliable surrogate. As an Agent of mathematics, it must prove capable in the abstract of performing calculations of all sorts. Now, calculations, like all rules, are necessarily ambiguous in their concrete formulations, always requiring specifications which can never be exhaustively formulated.[44] Calculation, then, becomes not just adhering to a fixed practice or process, but also agreeing on the correct result of the process.[45] It is, in fact, possible to formulate calculation solely in terms of agreement in outcomes.[46] In a certain sense, then, the Mathematician can trust the Agent because they both already know the answer. That the Mathematician never checks on the Agent after the thought experiment becomes properly mathematical would seem to further indicate that the Agent's conclusions are foregone. Moreover, these conclusions are foregone in a way that is prior to the implications of determinate deductive method. These conclusions condition and prefigure the outcomes of deduction, and are the very basis upon which such propositions come to have meaning. The Mathematician sends the Agent to verify an answer which must, by virtue of its capacity to be sent to the Agent, already be known and at hand.

Why, then, construct the Agent, who seems at this point to be nothing more than a convenient yes-man, as a powerful and versatile rule-follower? Moreover, how can what one would hope to be the independent agency propagated by strict adherence to a rule always manage to return the right answer? We can start with an answer to the second question, which more or less boils down to the observation that the statement or formulation of a rule and the outcome of its execution are in general only superficially related. This is because the meaning of a rule cannot be properly separated from the knowledge community within which the rule is articulated.[47] Rule statements are thus not in themselves determinate, but neither are their associated outcomes undetermined, for statements are but the visible part of the entire rule as such—the tip of the iceberg. A method, procedure, or proof is never

[43]Ibid., p. 12.
[44]Wittgenstein, 1972, I: 113.
[45]Lynch, 1992, p. 229, n. 18.
[46]Wittgenstein, 1972, I: 134.
[47]Lynch, 1992, pp. 222, 226. Lynch, 1992, "Reply," p. 289.

literally present in the individual details or material presentation of the argument.[48] We are necessarily blind to the specific network of understandings which allow us to correctly obey a rule,[49] and this fact makes rule statements, like statements in mathematics itself, seem far more deterministic than they actually are.[50]

The answer to the first question speaks to why we have rule statements in the first place. Put simply, it is because rule statements are a lot easier to believe in, comprehend, and accredit than rules themselves. There is a certain economy of form and meaning in a rigorously stated rule, which, we recall, is one which successfully elides the social circumstances and infrastructure of its production, articulation, and carrying-out. That nice, compact statements should correspond to nice, compact results makes perfect sense. Rules have been naturalized and incorporated into scientific culture to the point that reducibility to rules is often a dominant criterion for realism or scientific validity.[51] Rule statements provide a level of impersonality which effectively downplays the subjectivities at play in knowledge making.[52]

Impersonal rule statements suggest an underlying mechanical computability for a proposition, and are compelling in large part because they promise to separate results from their makers.[53] We can now recognize that rules are the mathematical implementation of the construct of mechanical objectivity discussed above. The turn-of-the-nineteenth century formalization of rigorous mathematics, whose origins can be traced to the work of Augustin-Louis Cauchy, relied heavily on mechanical objectivity's ideal separation of mathematical argument and production from individual human reason.[54] The most faithful implementation of this project has been the computer, whose intent as a calculating machine renders it autological with mathematical practice.[55] With such an emphasis on computability in view, it should not be surprising that computers have played important roles in especially complicated demonstrations in formal or quasi-formal mathematics.[56] Such associations come

[48]Livingston, 1999, p. 868.
[49]Bloor, 1992, p. 269.
[50]Bloor, 1991, p. 180.
[51]Porter, 1999, pp. 396, 398.
[52]Porter, 1995, p. xi.
[53]Porter, 1999, p. 401.
[54]Richards, 2006, p. 712. MacKenzie, 2004, p. 76.
[55]Rotman, 1997, p. 18.
[56]MacKenzie, 2004, p. 71. See also MacKenzie, 2001. In both publications, MacKenzie discusses in detail a number of issues arising from computer proofs, including testing, types

full circle in controversies such as those over gravity waves, where computers come to represent both mechanical objectivity and mathematical veracity for certain scientists.[57]

4.5 Proof as Testimony: Staging Mathematical Meaning

Having gestured at the most prominent mode of producing credibility, that of rule statement, we will now consider the broader structure of testimony among mathematicians. In particular, we shall examine the prover of a mathematical result in her capacity as a witness. Arguably, the principal point of a mathematical proof is to establish a matter of mathematical truth. As proofs are public, social productions, the matter of truth must be created through the assent of an audience.[58] As the audience also consists of members who are provers and producers of mathematics in their own right, we should simultaneously admit two more purposes to proofs: testing intuitions and aiding memory.[59] We will content ourselves here by noting that this hybrid conception of proof-making semiotically engages both the Person and the Mathematician, and shall keep this hybrid structure in mind as we unpack the testimonial structure of proving.[60]

As witness to a mathematical reality, a prover should fit well within our earlier discussions regarding the witness. The biggest difference is that the prover is particularly well adapted to act as a spokesperson,[61] and the allies the prover enlists include her imagined Agent and its entire Virtual Code of ideal mathematical operations and truths. The prover must, however, undergo a special regime of self-discipline in order to keep the Agent and Virtual Code on her side. Because of the logical structure of proving, the prover must constantly monitor the local accountability of her argument,

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of rigor, and the role of formal logicism in understanding computers. Computer proofs achieved a milestone of epistemic absurdity in an actual court proceeding over whether a particular microprocessor could produce mathematical proofs. MacKenzie, 2004, p. 73.

[57] Collins, 1985, p. 87.

[58] We discussed above (page 22) why this assent must be a compelled one, at least for Hobbes and Boyle. See Shapin, 1988, p. 32. See also MacKenzie, 2001, p. 11.

[59] Lakatos, 1979, p. 29.

[60] Rotman, 1988, pp. 6–7.

[61] Latour, 1987, p. 237.

checking and re-checking it for internal consistency and rationality.[62]

The prover is also in a funny position in her capacity as Person. While the human aspect of her work has to be excised in order for the work to be credible,[63] her credibility as a prover can be highly dependent on her personal situation. A case in point is the emergence in the early nineteenth century of the figure of the tragic mathematician. Around that time, worldly failure became closely associated with mathematical genius.[64] Narratives about how results were obtained lent credibility to results. In the case of geodetic mathematics, which involved expensive and arduous surveying missions conjoined to theoretical speculation and argumentation, the mathematical rigor and physical rigor in an expedition came to be associated with each other in narrative accounts.[65] This association comes both from explicit comparisons and from commonalities in the grammatical structure and terminology of parallel parts of the proof account.

A third peculiarity for provers as witnesses comes from the problem of replication.[66] Where juridical or scientific testimony typically seeks to narratively recreate the scene of production, the semiotic structure of mathematics both allows and requires the mathematical witness to fully replicate the result itself.[67] Math's essential non-materiality (or rather it's putative non-materiality) means that the material conditions of production are accessible to the witness's audience to the same extent to which they were present to the witness in the first place (that is, to no extent at all, provided one accepts the non-materiality conceit). The witness is no longer reviving a past event, but remaking, or replicating, that event as a current and original production. Where, in scientific testimony, replication and narrative recreation are distinct activities, in mathematics they are set up as one and the same.

When math is replicated in this way, by narrative recreation, its narratives become temporally ungrounded. In much the same way that the correct way of seeing a device is established only after it has been successfully replicated,[68] the full argument of a proof is only visible after it has

[62]Livingston, 2006, p. 51.

[63]c.f. Richards, 2006, p. 712.

[64]Alexander, 2006, p. 717.

[65]Terrall, 2006, p. 684.

[66]c.f. Shapin and Schaffer, pp. 229–230, 281–282.

[67]Livingston, 1999, p. 869 suggests these proofs may even be thought of as "rediscovered on subsequent occasions."

[68]Collins, 1985, p. 76.

been performed.[69] As in the usual story of scientific replication, when a proof is successfully replicated the prover is given license to write away the messy details of its production and re-articulation.[70] Even as a proof is being developed, there is a constant process of rewriting by the prover.[71] Such virulent revisionism is made possible by the putative stability of math's logical structure, which appears timeless and historically continuous by means of a backward appropriation of previous mathematics.[72] Thus, it is not only after, but during, and even before the proof begins that the right way of seeing it is reworked, refigured, and redeployed.

While an emerging proof narrative and the decisions of a prover go a long way toward determining the content and form of a completed proof, they do not tell the whole story. A proof is only ever sedimented and stable when it is accepted in one or more homologous forms by those who would cite or contest it.[73] Being written is no guarantee of acceptance, and being cited is no guarantee of being read.[74] What contestations and citations show, then, is the level of communal interest in an article or argument,[75] and so, in part, the comfort of those interested in the argument in making use of its results without re-deriving them.

Until a proof has stabilized—not necessarily coincident with its publication or first citation—community pressures can force adaptations and displacements which substantially alter the final appearance of a proof argument.[76] These changes needn't just be at the level of the particular argument, but can also affect the methodological principles of entire fields or subfields.[77] The field distinction is especially pertinent for mathematics, as its MetaCode is forbiddingly difficult and it is immensely difficult for outsiders to gain traction in any branch of the discipline.[78] Smaller scale communication and distribution systems come to dominate, and create their own promises

[69]Pickering and Stephanides, 1992, p. 143.

[70]Lakatos, 1979, p. 142–143. Glas, 1999, p. 3.

[71]Livingston, 2006, p. 60.

[72]Rotman, 1988, p. 33. See section 5.3.2 for the role of time and the construction of timelessness in mathematics.

[73]Thurston, 1994, p. 8.

[74]Ibid., p. 8.

[75]Latour, 1987, p. 39.

[76]Pickering and Stephanides, 1992, p. 146.

[77]Porter, 1999, p. 398.

[78]Rotman, 1988, p. 2.

and problems with respect to collaboration and standards maintenance.[79]

4.6 Matters of Principle

At the same time, in order to claim universality, mathematics must present itself as open to all rational humans (or at least all provers), *in principle*.[80] We recognize this claim in Boyle's opening of witnessing, through technologies of multiplication, in principle, to everyone.[81] Equal access to truth pervades idealist philosophies of science.[82] This unlimited extension of the mathematical knowledge community is supported by a conflation of the *in principle* with the *in practice* feasibility of an action. One can find this conflation throughout the Boylean sciences.

Rotman's model of mathematical semiosis calls into question the relation between the thinkable, writable, and sayable.[83] These invoke the interpenetration of the Virtual Code and Code, as mediated by the MetaCode. The writable, which is for Rotman the mathematical-semiotic version of the doable, is augmented by an infinitely vast realm of impossible operations legitimated almost purely by the 'in principle' conflation. But this conflation is just that: a conflation. Within the lived practice of mathematics, there is a large gap between that which is executable in fact and that which can only be done in principle.[84] This gap is not merely the one between finite and infinite calculations and operations. Even perfectly mobile mathematical inscriptions[85] can only be synoptically considered in relatively small groups.[86] Small measurements necessarily stand in for large ones.[87] Surveyable notation is allowed to stand in for the ideal objects of one's manipulations because the objects themselves are not manipulable.[88] In the face of human

[79] Jaffe, 2004, pp. 110–112, 114–115.

[80] Livingston, 2006, p. 51. For the witnessed aspect of this universalizing presentation, see Livingston, 1999, p. 874.

[81] Shapin and Schaffer, 1985, p. 25.

[82] Collins, 1985, p. 29.

[83] Rotman, 1997, p. 32.

[84] Rotman, 1988, p. 20.

[85] Latour, Visualisation, p. 10.

[86] Wittgenstein, 1972, II: 3.

[87] Ibid., II. 4.

[88] Wittgenstein, 1972, II: 16. For a discussion of surveyability in the constitution of a successful proof, see MacKenzie, 1999, p. 48.

contingencies of reckoning—fatigue,[89] error, even subterfuge—the 'in fact' is forced to stand in as an always inadequate substitute or proxy for what we would wish to do in principle, and the leap is always made by passing off our inadequacies to the Agent.[90]

The 'in principle' gesture is not limited to the ideal manipulations of the mathematical semiotic subject. In fact, some of the most interesting applications of the 'in principle' principle are in the Coded mathematics in the proof itself. Here, mathematical rigor is found as much or more in principle than in fact. Much of this work is done by the specter of a double translation of theorems into the metalanguage of pure symbolic logic and then back into vernacular mathematics—a specter which rarely, if ever, appears.[91] This imagined translation reduces proof to transcription by making all of the necessary onto-epistemological transformations accessible entirely in principle.[92] There can be gaps in this transcription, if the argument is made out to be rigorous, which can always be filled, in principle.[93] In principle translation needn't even operate only on already Coded mathematics. Boyle was always careful to argue that his results could be mathematized, while continually resisting their actual mathematization.[94]

Special force is accorded to the 'in principle' principle by its position within the calculable regime of mechanical objectivity. Leaving aside the problems we've already discussed with converting rules into mechanisms, we can see that in large computational systems the mechanisms themselves must invoke a certain suspension of concrete verifiability. We trust machines and we believe in their determinacy because their inner workings are, in principle, accessible.[95] In fact, however, the inner workings of calculating machines are

[89] Wittgenstein, 1972, II: 17.

[90] Rotman, 1988, p. 20.

[91] Rotman, 1997, p. 20. We shall discuss this translation in detail in chapter 5.

[92] Wittgenstein, 1972, III: 41. Specifically, Wittgenstein states that a proof must be reproducible by mere copying. Note my implicit distinction between 'accessible entirely in principle' and 'entirely accessible in principle'. While the latter is certainly a conceit of the logical paradigm (see chapter 5), the former expression captures Wittgenstein's sense of 'mere copying'—that is, the entire onto-epistemological work of proving is relegated to the realm of the 'in principle'.

[93] MacKenzie, 2004, p. 76.

[94] Shapin, 1988, p. 42. For a perspective on Descartes's seventeenth century project of mathematical physics and the mathematization it entailed, see Gaukroger, 1980, especially p. 98.

[95] Jordan and Lynch, 1992, p. 102.

largely forbidden to us by a range of technical obstructions,[96] including issues of surveyability discussed above.

Most of the time, the distinction between 'in fact' and 'in principle' does not matter.[97] The failure to distinguish the two is what grounds and legitimates the vast majority of mathematical practice. Save for in philosophy, they must be disentangled only when their conflated application is subject to dispute. Here, what is possible in principle is differentiated from what is possible in practice by the actual socio-material costs of dissenting.[98] That which must be left to the Agent is effectively out of bounds in a mathematical dispute, and the debate must be conducted purely in the space between metamathematical argumentation and properly practicable mathematical operations. It would appear that semiotic structures depend integrally on the linguistic and social structures which are the subjects of the next chapters.

4.7 Graphemes and Mathemes

Semioticians often begin to describe signifying practices by breaking them up into unitary instances of articulation. To their graphemes and phonemes, one can add the concept of mathemes.[99] By *matheme*, I mean a unit of mathematical articulation, typically situated within the Code of mathematical semiosis. These symbolic objects can be written, verbal, or imaginary, although there is certainly an identifiable alphabetic prejudice in mathematics, and particularly within the dominant Platonist mathematics.[100] No empty signifiers, mathemes play a creative and constitutive role in mathematical practice.[101] This can be said regardless of one's philosophical bias, whether formalist, privileging signs; intuitionist, privileging signifieds; or Platonist, privileging a regulatory ideal of independent truth.[102]

Where mathemes are contested or reflexively examined, they are generally so done on grounds of their correspondence with ideal (and legitimate)

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[96] Ibid., p. 103.

[97] Ibid., p. 104. A quote often attributed to Yogi Berra: "In theory there's no difference between theory and practice, but in practice there is."

[98] Latour, Visualisation, p. 17.

[99] There is some discussion of mathemes in the psychoanalytic literature which is tangential to the present argument.

[100] Rotman, 1997, p. 26.

[101] Rotman, 1988, p. 28.

[102] Ibid., pp. 4–5.

mathematical objects,[103] or of their representation of external or experimental reality.[104] For the early moderns, the question of proper semiosis and representation was particularly important, and problems were more often thought of in terms of mathematical tasks than in terms of mathematical truths.[105] Close analyses of semiotic practice reveal many ways in which it is in fact the ideal mathematical objects which arise through practices of representation.[106] Mathematical statements can also be modeled in terms of the action of the Agent, and are thus comprehensible as predictions or self-fulfilling prophecies.[107]

Invested with the proper truth-values, mathemes can neutralize themselves, rendering their own role in mathematical production invisible.[108] Language can thus be simultaneously seen as inadequate in its inability to capture a complete ideal mathematical image and over-adequate in its ability to generate mathematical absurdities, and thereby acquires an epiphenomenal position with respect to mathematical truth.[109] Stripping language from math, philosophers can remove other undesirable contaminants, such as psychology, as well.[110]

One can't get far in this analysis without attending to the complexities of symbolically rigorous method. While most mathematicians are Platonists when it comes to their objects of study, most mathematicians subscribe to one brand or another of logical formalism when their methods are in question. Vernacular mathematics must be disassembled and reassembled in terms of right logical method in order to cement its onto-epistemological status as right knowledge.[111] This is a prime example of a transformation which hap-

[103]Porter, 1995, p. 11. MacKenzie, 2004, p. 72.

[104]Shapin, 1988, pp. 23, 46–47.

[105]Bos, 2004, p. 63.

[106]Rotman, 1988, p. 31.

[107]Ibid., p. 13.

[108]Rotman, 1997, p. 19.

[109]Hacking, 1999, p. 166. For Platonists and intuitionists, this excusal of language is clear. For formalists, it happens by way of a foundational reclaiming of language through a highly dramatized originary axiomatization. A canonical example of the valorization of definitional starting points in geometric philosophy is Hobbes's model of proper knowledge. Shapin and Schaffer, 1985, p. 100. This approach posits an Archimedean point which, by acknowledging the foundational flexibility of axiomatic starting points, contains in its very enunciation its own ultimate undoing. Bloor, 1991, p. 18.

[110]Daston, 1999, p. 117.

[111]Rotman, 1997, p. 20. Bloor, 1978, p. 269. Incidentally, Hilbert, the late nineteenth century mathematician who worked to extend the program of formal rigor across

pens almost exclusively in principle and almost never in practice, save for a handful of textbook or hand-picked examples. Compounding the difficulties inherent to such translations when they do happen is the necessary inexactitude required to formulate a vernacular or geometric proposition rigorously.[112] Logicism always entails a positioning with respect to the sign-value of mathematical propositions.

While logical translations remain mostly hypothetical, mathemes are typographically translated at nearly every juncture, forming highly heterogeneous symbolically and semiotically rich texts.[113] Mathematical texts carry a double weight. They are, on the one hand, the physically manifested accumulations of signs vaunted by formalist rigorous practice. On the other hand, they are the very linguistic elements which are written over by Platonist onto-epistemologies. Such texts are sacred, but they are not integral. Thus, rigorous arguments (which do not necessarily here include formalist presentations of pure sign manipulations) are generally highly surveyable, and robust in the face of typographical mistakes.[114] As Hardy put it, theorems in geometry are not affected by the quality of the drawings.[115]

4.8 Diagrams and the Sesquitext

We close by turning our attention to Hardy's drawings, for they are the most literal materializations of mathematical witnessing. In figures and diagrams,

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the subdisciplines of mathematics, had only intended his logical formalism as a form of metamathematics, not as an integral part of the mathematical Code itself. MacKenzie, 2004, p. 81.

[112] Derrida, 1978, p. 162.

[113] Rotman, 1988, p. 6–7. Latour, 1987, pp. 238–239.

[114] One Cornell University mathematics professor certified to me that good mathematical papers were 'correct modulo typos,' that is, correct if one allows for the virtual identification and correction of any purely typographic, as opposed to mathematical, mistakes. Mathematics is not alone among the disciplines in having means of excusing the representational shortcomings of a rigorous presentation, but is perhaps special in that the epistemic foundations of the discipline's rigor are from the outset more rooted in the presentation itself than, say, for observational sciences.

[115] Hardy, 1967, p. 125. Livingston, 1986 and 2006, explains that this is hardly the case in practice, although it remains a central assumption in theory. Netz, 1999, pp. 33–34 discusses Poincaré's characterization of geometry as reasoning about poorly drawn figures, which for Poincaré meant figures which could be drawn by anybody and still retain their mathematical potency.

we have a literal visual image being regarded as both a mathematical object as such, and as a representative of an entire class of mathematical objects. Hardy's model of mathematical drawing is drawn from the classical proofs of Euclidean geometry.[116] There, a single diagrammatic triangle, circle, or line is both the specific local site of mathematical operation and the faceless stand-in for the whole range of possible figures of its type. I propose to call this dual structure *sesquitextuality*.[117] As we shall see, this structure is always in practice not quite dual, hence the prefix *sesqui*, for one and a half.

Sesquitextuality encompasses a wide range of mathematical semiotic activity. With respect to diagrams, mathemes, logical statements, variables, or any of a number of mathematical sign-productions, each figure has *both* a literal formal existence, in itself, on the page or chalkboard, which is a direct object of mathematical rhetoric, *and* a metonymic existence, as a spectral representation of countless infinities of possible object-type-examples within the Virtual Code.[118] What is most important is that while this latter manifestation is crucial to the onto-epistemological status of mathematics, it is never made to show its hand. As we have argued, the Virtual Code enters into mathematics precisely insofar as it is cited without being seen. It is an always deferred half text which permeates the primary formal text of the figure itself.

To draw out the role of sesquitextuality in witnessing, consider the example of the diagram. First and foremost, diagrams stand in as the objects of mathematical proofs. These diagrams, which are always interpretively flexible,[119] delimit the objects of a proof through their constitutive exemplarity. The choice of figure can drastically alter both the course of a proof[120] and

[116]The role of diagrams as emergent metonymic elements of Greek mathematics is discussed extensively in Netz, 1999. Høyrup, 1996, explores the constructed narrative links fabricating the connection between Greek mathematics and modern European mathematics.

[117]This coinage borrows in part from the structuralist tradition of playing with prefixes to evoke different textual relationships. See, for example, Genette, 1997.

[118]There is a related semiotic distinction between the individually manifested word and the vast expanse of language it invokes. I would maintain that sesquitextuality as it appears in mathematics has important differences from this general situation of language, particularly when it comes to the putative work done by the imagined figure of the Agent, although a theory of sesquitextuality may well enrich the general project of semiotics.

[119]Latour, Visualisation, p. 17.

[120]Livingston, 2006, p. 40.

the domain of its application.[121] As an exemplar, the diagram lends a precise visual clarity to what can otherwise be obtuse or incomprehensible symbolic formations.[122] Within its position as a materially manifested exemplar, the diagram is a workable object within the proof, and must be subject to tangible manipulations on behalf of that for which it is a proxy. All this, despite the diagram's putative externality to the proof procedure, which must, after all, claim to go far beyond the individual diagram in its implications.

In math, as in Boylean science, the diagram is what the audience of a piece of testimony has in mind during the narrative. It is a stand-in for observation. For this reason, Boyle and Hooke commissioned expensive naturalistic engravings for their reports of air-pump trials.[123] It is also for this reason that Hobbes's figures, where they appear at all, are purely geometrical.[124] For each, the figure models the type of vision he demands of his audience. It matters not that such diagrams were, in general, unreliable,[125] but that they were able to stand in for actual material experimental practices—for all material experimental practices, in fact, which can be called replications of the experiment at hand—and could take on the status of witnessed object for those far away from the immediate presence of the putative experiment.[126]

Within the regime of sesquitextuality, witnessing is refigured as a practice of suggestive representation. Its narratives function by invoking archetypal or representative examples and donating to them the force of a vast array of virtual cases. But the power of witnessing is precisely in the ability of the virtual cases to remain virtual—for proper replication to remain impossible, for operations on entire classes of ideal objects to remain inaccessible—without compromising the force of the testimony. An inscription stabilizes and mobilizes only insofar as it controls its exemplars and only insofar as its exemplars control the space of representational possibility. Witnessing produces the force of science and mathematics precisely by virtue of the sesquitext.

The represented objects of mathematics are thus always almost already manifested, and always essentially out of reach. As such, mathematics can be the ideal subject of Platonists, the concrete subject of formalists, the imaginative subject of intuitionists, and virtually any other type of subject

[121] Wilder, 1961, pp. 17–18.
[122] Livingston, 2006, p. 57.
[123] Shapin and Schaffer, 1985, pp. 32, 61.
[124] Ibid., p. 146.
[125] Ibid., p. 262.
[126] Latour and Woolgar, 1986, p. 51.

for virtually any other type of mathematician. The displacement of the sesquitext creates a gap within which competing mathematics can flourish. This gap is present, as we have seen and will continue to elaborate, at all levels of proof-making, ranging from the formal presentation of a proof to the abstract realm of mathematics in which the proof can claim to operate. Mathematics is practiced, therefore mathematics is sesquitextual.

Chapter 5

Witnessing in Mathematics: A Translational Model

The end of the nineteenth century bore witness to what might be called the great revolution in symbolic logic. That era's great accomplishment was to reduce the rigors and requirements of human reason to the formal manipulation of words and symbols on a printed page. If this description sounds parodic, it only serves to underscore the grand visions shared by the likes of Weierstrass, Dedekind, Kronecker, Hilbert, and, perhaps the last of the great logical messianists, Bertrand Russell.

We saw in chapter 3 how Cauchy's turn of the century work cleared the way for formal algebraic manipulation to be taken as a foundation of the entire project and corpus of mathematics. By admitting the possibility that completely abstract symbols from algebra could be accorded the status of rigorous elements of a proper proof by means of a few limiting assumptions and some methodological discipline, Cauchy established the basic framework of the symbolic calculus. Mathematics was to be based on these newly stabilized, controlled, and mastered algebraic forms.

Such algebraic forms were to be clearly specified, in good Euclidean style, from the beginning of the argument. In Russell's propositional calculus, they were to be simultaneously completely general and completely particular. Russell's algebraic forms were completely general in that an abstract proposition (say, the proposition p) could stand in for an infinite variety of concrete propositions (for instance, 'Socrates is mortal', 'all men are mortal', or 'all men are Socrates'), whose truth value could be decided by means of a completely general and abstract decision procedure. These forms were also completely particular, in that they were to be precisely and concretely fixed at the locus of production: the mathematical text. Russell's vocabulary needed to be bounded and closely guarded.

Logical propositions were to be validated by means of a concrete and fixed system of manipulations, which were to be policed by limiting assumptions in the spirit of Cauchy. Half of logic is foundational propositions, and the other half is the set of rules for creating new propositions from these primordial elements. The logical messianists of the end of the nineteenth century were sure that this system could rescue mathematics from itself by endowing it with a new and absolutely unimpeachable rigor. They were also sure that all of mathematics, or at least all of the mathematics worth preserving, could be translated into the terms of this solid foundational system.

This chapter is about translation, logic, and the possibility of mathematics. Where the previous chapter discussed mathematics and mathematical discourse in terms of the implied, understood, or deferred meanings of mathematical statements, this chapter will look at mathematics as a total linguistic system. I shall explore the assumptions and practices undergirding this system, as well as its implications for the structure and function of mathematics as a discipline. Where the previous chapter was about the meaning of mathematical proofs, this chapter is about the meaning of mathematics.

5.1 Russell the Visionary

Symbolic logic, even after the foundational rupture initiated by Gödel in the interwar period of the early twentieth century, remains emblematic of the relatively unquestioned triumph of a remarkable space of translation in mathematics. Confidence in logic at the turn of the twentieth century was so high that Bertrand Russell, at the outset of his 1903 book *Principles of Mathematics*, was able to declare:

> By the help of ten principles of deduction and ten other premisses of a general logical nature [given by Professor Peano]. . . , all mathematics can be strictly and formally deduced; and all the entities that occur in mathematics can be defined in terms of those that occur in the above twenty premises. . . . The fact that all Mathematics is Symbolic Logic is one of the greatest discoveries of our age; and when this fact has been established, the remainder of

> the principles of mathematics consists in the analysis of Symbolic Logic itself.[1]

After millennia of fumbling in the dark with imperfect systems of proof, it finally seemed to Russell that mathematicians had access to the ideal forms dancing outside of Plato's cave. Math was to be conceived of as a system of propositions, and mathematical logic was to provide concrete, perfectly discernible, and fundamentally irrefutable rules for relating one proposition to another. Every true proposition could be reduced to a tautology. Every false proposition could be reduced to a contradiction. Everything could be known, and everything could be mastered. There was to be no more room for methodology. Debates over how best to do mathematics, and how best to certify mathematical truths, could now be considered settled because everything was to be reducible to some combination of Peano's twenty principles.

If Russell was right, how is it that after decades of logical evangelism by him and his contemporaries all mathematicians aren't logicians? An immediate answer is that even if mathematics *truly* works as Russell claims it does, that is not how mathematics is used, practiced, developed, explained, or understood. Symbolic logic is an eminently difficult and complex idiom for producing useful mathematical tools and formulae. It elides the fact that, to borrow from Wittgenstein, mathematics "is usable, and, above all, *it is used*."[2] We might be tempted to ask an entirely different question: if logic is useless, why is it studied? By all rights, there should be no one studying logic today. Russell should be a laughing stock: a misguided utopian buffoon.

Nonetheless, symbolic logic remains an active and thriving subfield of mathematics today. Russell's premises derive from a manner of thinking which has a much more pervasive grip on mathematics and the theory thereof than his prescription for mathematical practice could ever have. What we will

[1]Russell, 1937, p. 4. I should note here that the Russell I quote from *The Principles of Mathematics* represents a very small slice of his thinking, which had already undergone noticeable changes by his and Alfred North Whitehead's 1910 *Principia Mathematica*. See Monk, 1999. I should also note that I take Russell to be *symptomatic* of, not central to, the program of rigor as it was refigured in the late nineteenth and early twentieth centuries. His work will stand in, here, for a body of scholarship by those discussed briefly in chapter 3, among many others, far too broad for the space of this analysis. Russell was neither the first nor the last to attempt a theory of universal mathematics. See Schuster, 1980, for Descartes's often overlooked attempt.

[2]Wittgenstein, 1956, I: 5.

term the *logical paradigm*[3] has formed the core of mathematical thought at least since the end of the nineteenth century, with precursors in Cauchy and, earlier, in the Euclidean mathematics of the early moderns. This paradigm, as I will argue, is ultimately a potent theory of translation, and one which can be effectively analyzed within a well-adapted critical framework from the study of translation and national languages.

5.2 Math as a Language

Before describing mathematics and the logical paradigm as a linguistic system, a brief justification is in order for the project of studying mathematics as a language in the first place. On the one hand, it might be said that, since it is a linguistically communicated and comprehended practice, it is impossible to study mathematics except as a language. Yet, somehow, and with sparing exception, even social studies of mathematics, not to mention Whiggish logical syntheses of the discipline, tend more often than not to take the linguistic status of mathematics for granted.

This taking-for-granted has two common forms. The first is to make the logician's mistake of treating mathematics as an utterly singular language where all words and grammars are both perfectly unambiguous and, just in case words do interfere or interject themselves into the practice, also epiphenomenal to the real work of mathematics. For adherents to the logical paradigm, mathematics is linguistic only in the barest sense of using words to convey meaning, but without all the subtleties, socialities, and subversions of a real language. The second form, just as pernicious but often harder to single out, is the ethnographer, historian, or sociologist's mistake of treating mathematics just like any other language.[4] While mathematics is, fundamentally, a communicative discourse, it is grossly misleading to say that it functions like any other language, particularly any so-called *natural language* which permits a parlance in the context of the everyday. Mathematics is

[3]Again, we shall see how symbolic logic carries the characteristics of a Kuhnian paradigm. Kuhn, 1996.

[4]Bloor, 1976 and 1978, might be most obviously susceptible to this criticism, although he can be forgiven some overstatement in trying to open mathematics up to sociology. His argument amounts to claiming that mathematics can be studied *just like any other social system*, implicitly including those with more vernacularly-rooted vocabularies and conceptual frameworks. The fact remains that while all mathematicians are humans, mathematical interactions cannot strictly be viewed like any other human interactions.

a special language, with special conditions of existence, special uses, and special meanings.

A valuable truism to remember is that a language is constituted by its community of speakers. This is so regardless of the putative independence of the language with respect to this community. Derrida strengthens this thesis by asserting that "Every culture institutes itself through the unilateral imposition of some 'politics' of language," adding that "Mastery begins... through the power of naming, of imposing and legitimating appellations."[5] Under this formulation, language is precisely the means by which a culture asserts its culturehood. Moreover, this assertion is a unilateral imposition and exercise of power which is expressed primarily through naming. Derrida's explanation resonates well with critiques of language and nomenclature derived from varying brands of identity politics, but if read at face value it seems to miss a certain underlying structural hegemony in the politics of language.[6]

A language is not just the sum of its terms. Countless semioticians over the last century have said as much. There is certainly a potent politics of naming, but there is an even more pervasive and essentially linguistic politics of combining. The politics of combination dictate which words can be used when, where, and how. They encompass both syntagmatic and paradigmatic realms of semiosis. They operate at the level of both *langue*, the formal structure of language, and *parole*, the spoken practice of language. Such structural politics are evident, it might be argued, no more so than in rigorous mathematical presentation, where structural rules give truth-making power to a mathematician working with any combination of named concepts. Before Cauchy, one could be justified in concluding that the central work of mathematics was in making and naming concepts to fit the sundry problems to which mathematics was applied. After Cauchy, there can be no mistaking the fundamental position of mathematical structural rigor at the heart of mathematical production and practice.

Derrida's formulation of the power of naming is amenable to a more Wittgensteinian view of language as a *system of communication*,[7] which might better reconcile naming with the *gestalt* picture of language politics just described. Complicating the Augustinian description of language as a pure system of signifiers and signifieds with a strict referential correspon-

[5]Derrida, 1998, p. 39.

[6]Derrida is of course cognizant of the structural politics of language, within which naming operates.

[7]Wittgenstein, 1963, I: 3.

dence, Wittgenstein argues that language must instead be comprehended as a communicative system based on learned practices of meaning-making.[8] In this light, the power of naming has two functions. First, it stabilizes that which is named within a specific context. To use Wittgenstein's example, naming 'red apples' attests to an essential 'apple-ness' which can be found in certain objects at a store or market, and assigns a unifying normative rubric under which other things can be called red apples than just what someone requesting 'red apples' would have in mind.[9] Naming implies an annexation and an exclusion through the process of signification embodied in meaning-making, as nomenclature gives mastery over the named objects.

Second, it mobilizes an object to serve as knowledge currency. Wittgenstein cannot write about 'red apples' without imposing such a name and such a context of reference. His use of the phrase, in terms of knowledge work, does not fit nicely within the theory of informal learned referential practice he describes. All of this name work functions both to master objects and to place them within a structural politics of language as a system. Derrida's *legitimating appellations* are ones which affix not only names as such to objects in isolation, but also systems of structure and division on the entire space of potential objects or object potentialities. Names are not names in isolation.

5.2.1 Logic, Pure Language, and Math's Benjaminian Promise

An article from 25 December 1959 in the journal *Science* speculated that if scientists were to make radio contacts with inhabitants of other planets they should send pulses to communicate prime numbers. Lancelot Hogben, a popular writer on mathematics, explained to *Time* on 14 April 1952 that "earthlings" should begin conversation with "their Extra-terrestrial Neighbors" with "some small talk about numbers, whose properties do not vary from planet to planet."[10] This sentiment informed Jodie Foster's declaration in the 1997 movie *Contact*, based on a novel by Carl Sagan, that "Mathemat-

[8] Ibid., I: 1–3.

[9] The threat of impostors is a constant one in language and in logic. Lakatos, 1979, gives many examples of the mobilization of language to definitionally bound a set of mathematical objects, and shows how definitional impropriety always manages to reassert itself.

[10] Wilder, 1973, p. 34.

ics is the only true universal language."[11] The popular scientific imagination is full of references to mathematics as a universal language which is independent of context or circumstance. It is presented as an idiom intelligible to not only every human, but every intelligent life form, real or imagined, present or impossibly distant. Indeed, mathematics may well even be a criterion by which intelligence is judged by classifiers of one stripe or another.

As a so-imagined universal language, mathematics need not be capable of saying everything that any other language can. It is hard for even the most optimistic idealist to imagine the whole range of human experience and the entire possibility of language to be expressible mathematically. A better way to understand the promise seen in mathematics is through the notion of a *pure language*. Pure language is an idea quite distinct from universal language, with the latter often appearing as an attempt to reclaim a sort of Babelian unity or linguistic transcendence. Instead, pure language refers to that which forms the base which makes possible all systems of language.[12] Put another way, it concerns "the being-language of the language."[13]

The concept of pure language was developed in Walter Benjamin's essay "The Task of the Translator." There, he argues that all languages are interrelated in what they want to express.[14] This fabrication of subjective intention in an abstract language creates a certain kinship, which is not to be confused with resemblance, among all languages.[15] Because all languages intend to express the same thing, translation enriches all languages involved by introducing new means of expression.[16] Pure language would then be the totality of all languages and their intentions supplementing each other to access a universal object of expression.[17] Yet pure language itself is not expressive—it does not operate as a part of everyday communication—but is rather that which is meant in all languages, and so is instead already present

[11]Internet Movie Database.

[12]Morinaka, 2006, p. 2.

[13]Derrida, 2002, p. 131.

[14]Benjamin, 1968, p. 72.

[15]Derrida, 2002, p. 130. There is some subtlety here which is brought into relief by considering Wittgenstein's notion of *family resemblance*. All languages, in the Benjaminian model, are related by what they want to say, but not only can you not give any uniform point of similarity between all languages, which would be Wittgenstein's point, but it is further possible that two languages would appear entirely incommensurable, but for the purported intentional commonality which accords them kinship.

[16]Benjamin, 1968, p. 73.

[17]Ibid., p. 74.

in every use of language.[18] There is a promise carried in the *vouloir-dire*, or meaning-to-say, of each language which makes pure language possible.[19]

Mathematics is imagined as part of the base of common ideas which each language means to say.[20] This is why anthropologists can claim that systems of counting and numbers are cultural universals, found in ethnographies of every known society.[21] Of course, the fact that numbers are found in every culture is as much a reflection of who is looking as of who is being observed. We have described in prior chapters how the early moderns, with Hobbes a prime example, were utterly fascinated with their interpretation of Euclid's schema for mathematics, and this led to the adoption of axiomatic geometry, an important predecessor to symbolic logic, as a widely regarded model of philosophy in general.[22] Philosophy, in turn, molds the categories of observation and classification used across the scholarly disciplines, and has done so in the Western imagination at least since Aristotle. A valorized straw figure of Euclidean mathematics permeates the whole of Western intellectual and cultural production in ways which fundamentally shape the imaginative and enunciative possibilities therein.

The promise of pure language, by invoking the totality of all language, necessarily invokes as well something which is independent of language. What all languages would say if they could would be something exterior to each language in itself, and the totality of language itself, as well. Mathematics is an especially plausible candidate, model, and exemplar for pure language because most people, mathematicians and non-mathematicians alike (although in different ways and often for different reasons), tend to believe that there is a natural underlying basis for mathematics. While many mathematical concepts, such as complex numbers, begin as abstractions and curiosities and later find their way into the natural sciences and our understandings of nature, there is an underlying belief that things such as numbers, arithmetic, and geometry correspond to and are rooted in fundamental properties of reality, which may or may not be exterior to the material world. There

[18]Ibid., p. 80.

[19]Derrida, 1998, p. 67.

[20]Benjamin alludes in an fragment penned just before "Task of the Translator" to a sort of pure knowledge which can only be understood through the relations between language and logic, so it is entirely possible that he would approve of this attempt at understanding mathematical logic as a sort of pure language. Benjamin, 2004, p. 274.

[21]Wilder, 1973, p. 32.

[22]c.f. Shapin and Schaffer, 1985, pp. 100-101.

would, for instance, be an abstract, natural, and ideal version of a number, or a line, or a polygon, to which mathematical statements refer. This view can be called mathematical Platonism, and a central goal in the sociology of scientific knowledge has been to chip away at this ideal's foundations.[23]

Bloor asks his readers to think of the terms and meanings of mathematics by giving the example of a corresponding idealization of the notion of *hat* in a young language learner. A series of misfires and adaptations demonstrates for Bloor that an ideal notion of hat comes from successive encounters and socializations rather than a derivation of essential *hat-ness* from the learner's natural context.[24] The implication is that any attempt at defining just what a hat is would always be after the fact, and moreover it would always be helplessly incomplete. We have already seen this difficulty when it comes to formulating and understanding rule statements and logical instructions. Russell's program would seem doubly challenged: he can neither give a clear idea about what he's talking about, nor can he fully or adequately explain what he's doing. Clearly, the lasting successes of Russell's program would indicate that more is at play than is revealed in Bloor's simple illustration, and it is precisely math's status with respect to pure language which may account for much of the gap between Russell and Bloor, as we shall see.

One immediate counter to Bloor's example illustrates how both rigorous mathematics and pure language inhabit a linguistic ideality which may often elide language itself. As I observed on page 93, mathematics puts itself outside of language by making the descriptions of mathematical objects exterior to the putative objects themselves. Thus, a failure to articulate an essential hat-ness is a shortcoming of language, not of representational possibility. There may actually be an essential, pure, ideal way of telling what a hat is. It is just that we are incapable of articulating it. It would be found, instead, in pure language. Bloor would not be alone in retorting that if it is inarticulable, it may as well not exist. There is no responding to a Platonist who insists on the existence of something purely inert.

Of course, something may be completely inert without the concept of or belief in that thing being empty as well. One need turn only to theology, and in particular the Kantian conception of God which argues that while any

[23]Bloor, 1978, p. 248. Related is the question of whether new mathematics is invented or discovered. Wittgenstein's view that, ultimately, "The mathematician is an inventor, not a discoverer" (Wittgenstein, 1956, I: 167) remains rare in the common imagination of mathematics.

[24]Bloor, 1976, p. 139.

attempt to settle the question of the *actual* existence of God is ill-founded, a society still does well to consider the social effect of actually materializable *beliefs* in God. No atheist would argue that belief in God has no significant effects. We can certainly study mathematics from the angle of the effects of *adherence* to Platonist philosophy while remaining entirely neutral to the question of the ultimate *correctness* of such a philosophy.

But the symbolic calculus made possible by Cauchy and brought to fruition over the subsequent century did more than allude to a distant ideal of mathematical form. It actually materialized the hitherto Platonic foundations of the discipline. Russell gave foundational status to the material graphical elements of his propositional calculus. Peano's twenty axioms were not just abstract starting points. They were manipulable, stable, mobile, and combinable materializations in the form of concrete graphemes on a printed page. In symbolic logic, the symbols are the foundations. The sign is prior to the signified. Mathematics offers a special window into the problem of Platonism and language because, formulated in just the right way, the two become one and the same.

5.2.2 The Logical Pivot

When Russell writes about mathematical ideality, one must not forget the particular tradition of ideal figuration in which he takes part. As Derrida puts it, "One should never pass over in silence the question of the tongue in which the question of the tongue is raised."[25] Every language implicitly carries with it a figurative paradigm which creates and decides the limits of speech and ideation, albeit rarely in a determinate way. It is fitting that questions of the decidability of logic are most often answered from within logic itself, or at least with the Platonic ideal of mathematics in mind.[26] The classic example is Gödel's logical proof of the incompleteness of any logical system sufficiently complex to include arithmetic. Logic becomes a pivot language in which all mathematics is couched.

Despite aspirations to the contrary, pivot languages are by no means value-neutral.[27] It is thus quite consequential that symbolic logic positions itself as the arbiter or decider of mathematics. It is not that no other mathematical systems exist, but rather that every mathematical system is regarded

[25]Derrida, 2002, p. 104.
[26]Wittgenstein, 1956, Appendix I, V: 27.
[27]Morinaka, 2006, p. 8.

as a coarse form of inquiry which is perfectly translatable into and reducible to symbolic logic. Logic acquires a referential ubiquity against which all other mathematical discourses are weighed, in the same manner as does the mythic West, another system fabricated out of oppositional idealizations.[28] The term *common denominator* has been borrowed extensively from mathematics to describe systems such as logic which purport to make all other systems commensurable. In this sense, logic within mathematics can be compared to English within international discourse.[29] All mathematics can be said to be capable of passing through logic in order to meet all other mathematics.

This measuring of mathematical discourse against logic effaces the peculiarities of particular sub-disciplines of mathematics. Analysis and Topology textbooks frequently make reference to the logical Axiom of Choice, and the possibility of logical decidability pervades the subtext of most specialized works. Even papers in so-called experimental mathematics are expected to provide provable algorithms in order to accord a measure of logical certainty to their results. These varied mathematical practices all must 'lose their accent' in order to enter proper mathematics, to paraphrase Derrida.[30]

As a result of English's position as a pivot language today, new works must presume, if they are to be of significance, that they will be translated into English, regardless of the languages spoken by their authors.[31] A similar phenomenon is found in mathematics, but with a crucial difference. A joke is often recounted where a mathematician is shown into a room with a fire burning in the center and a bucket of water sitting next to the fire. The mathematician picks up the bucket and puts out the fire. Next, the mathematician is shown to an identical room with a fire in the same location, but with the bucket of water sitting on a chair. The mathematician takes the bucket off the chair, puts it on the floor next to the fire, and walks away, having reduced the problem to one which has already been solved.

The structure of logical mathematics goes beyond the natural sciences in making reproducibility an *unstated* and *unproblematized* assumption. Strik-

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[28]Traces, 2006, p. 3.

[29]Ibid., p. 43. This analogy should not be construed as implying that logic is like English, nor that mathematics is just another manifestation of international discourse. We shall not have the space to discuss the spread of the logical paradigm, which may be similarly likened, although with important differences, to English's imperial-hegemonic diffusion.

[30]Derrida, 1998, p. 45.

[31]Traces, 2006, p. 42.

ingly many mathematical proofs consist of taking one problem and expressing it in terms of one or more problems which have already been solved. Thus, for the mathematician, it is assumed of works of significance not that they *will* be translated into symbolic logic, but that they *can* be so translated. In chapter 4, we called this phenomenon sesquitextuality. Here, sesquitextuality appears as a meta-linguistic phenomenon governing the space of practiced and implicit translation. There is an added assumption in mathematics that logic has already been used to justify the basic principles and tools of each mathematical specialty, which is part of the project of Russell's early twentieth century work.[32]

5.2.3 The Logical Structure of Mathematical Narrative

Attentive to the question of the logic in which the question of logic is formulated, I shall now describe the quasi-Euclidean logical system championed by Russell in terms drawn from the set theoretic functional calculus which is characteristic of much of contemporary mathematics. The goal of this section is three-part. First, I aim to provide a descriptive resource in an alternate vocabulary to aid in understanding the rich narrative complexity of the model of proof dominant by the end of the nineteenth century. Second, I aim to give a non-mathematician readership a glimpse of the form taken by attempts to translate common notions into mathematically acceptable formulations.[33] Third, I hope to expose the ways in which Russell's program is ideologically and structurally dependent on a system of propositional logic which is by no means natural, inevitable, or prior to the work of mathematization.

Let us begin by defining a mathematical *work* as a set (P, d, c) with the following components:

- P is a sequence of propositions $\{p_0, p_1, p_2, \ldots\} = \{p_i\}$ with a *narrative ordering*, i.e. ordered according to the order in which they appear in the work.
- $d : P \to 2^P$ is the logical dependency function. It maps a proposition to the subset of the entire sequence of propositions on which the given

[32]Russell, 1937. Russell and Whitehead, 1950. These are both later editions of works first published, respectively, in 1903 and 1910.

[33]Livingston, 1999, pp. 877 and 880 attempts a definition of the term *a-proof-description* which can be read in this light.

proposition depends in order to be true. We say a proposition p is *logically dependent* on a proposition q if $q \in d(p)$. Logical dependency is generally an *implicit* property of propositions, as it will generally not be explicitly described in the work.[34]

- $c : P \to 2^P$ is the internal citation function. It maps a proposition to the subset of the entire sequence of propositions which consists of those propositions cited in the proof of the given proposition. We say a proposition p *cites* a proposition q if $q \in c(p)$. Citation is generally an *explicit* property of propositions, as citations occur immediately following the proposition in the proposition's proof.

For the function d and a set of propositions $P_0 \subseteq P$, we define $d(P_0) = \cup_{p \in P_0} d(p)$ to be the union of sets on which propositions in P_0 are dependent. We may thus inductively define $d^i(P)$ as $d(d^{i-1}(p))$ for $i \geq 2$ and $d^1(p) = d(p)$. Similarly, $c(P_0) = \cup_{p \in P_0} c(p)$ and $c^i(p) = c(c^{i-1}(p))$ for $i \geq 2$ and $c^1(p) = c(p)$.

We call a work a *logical work* if it satisfies the following two conditions:

- $\forall p \in P$ and $\forall i \in \mathbf{N}$, $d^i(p) \subseteq d(p)$, and
- if $i \leq j$ then $p_j \notin d(p_i)$. (Alternatively stated, if $p_i \in d(p_j)$ then $i < j$.)

The first condition says that if a proposition p depends on a proposition q, then p also depends on all of the propositions on which q depends. The second condition says that if one proposition appears before another in the work, then the former cannot depend on the latter, nor can a proposition depend on itself. Often, in order to motivate a result, textbooks and papers will violate the second condition by placing the statement of a major result before statements of lemmas on which the result depends. One workaround is to treat the narrative ordering by the order in which the proofs of each result are completed, in which case virtually every professional mathematics publication in the last century and a half sets itself up as a logical work. The other workaround is to define a *logically viable work* as a work for which there exists a narrative ordering on P for which the work is a logical work.

[34]Davis, 1995, distinguishes between semantic and syntactic logical dependence. This distinction arises out of some structural considerations from the development of logical proofs, and a particular consideration of Gödel's theorem. For the purposes of this paper, it suffices to note that logical dependence takes many different *logical* forms, as well as the different *narrative* forms I shall discuss.

This is not an issue in Russell and Whitehead's *Principia*[35] because their presentation follows the strict order of proposition, proof, next proposition, next proof, and so forth. For the remainder of this section, we shall consider only logical works.

We are now prepared to offer our first structural lemma:

Lemma. The relation p *depends upon* q, denoted $q \leq_d p$ and defined by $p \leq_d p$ and $q \leq_d p$ if $q \in d(p)$, forms a partial ordering on P in a logical work.

Proof. We have by definition that $p \leq_d p$. If $p \neq q$ then we cannot have $p \leq_d q$ and $q \leq_d p$ by the second condition for logical works.

Now, suppose $r \leq_d q$ and $q \leq_d p$ with $r \neq q \neq p$. Then, by definition, $r \in d(q)$ and $q \in d(p)$. Thus, by the first condition for logical works, $r \in d^2(p) \subseteq d(p)$, so $r \in d(p)$ and hence $r \leq_d p$. Therefore, $\leq_d$ gives a partial ordering on P. This completes the proof.

By construction, the map from P sending each element to its numerical position in the narrative order on P is an order-preserving map with respect to the partial ordering $\leq_d$. We can say more about this poset. Define the *weight* $w(p)$ of a proposition $p \in P$ inductively as follows. If $d(p) = \emptyset$, set $w(p) = 0$. If $d(p) \neq \emptyset$, set $w(p)$ to be $1 + \max_{q \in d(p)} w(q)$. We know since we have a logical work that $d(p_0) = \emptyset$ for p_0 the narratively first proposition in P, and each subsequent proposition $p \in P$ has either $d(p)$ consisting of narratively prior propositions or $d(p) = \emptyset$. Every chain in P under the $\leq_d$ ordering is of strictly increasing weight, but it is not in general true that P is ranked by the weight function under the ordering $\leq_d$. Figure 5.1, from Russell and Whitehead's *Principia*, contains many examples of maximal chains of different lengths between the same two propositions. Call the set of propositions $p \in P$ with $w(p) = 0$ *axioms* or *fundamental propositions*, as they do not depend at all on prior propositions.

For a given proposition $p \in P$, the *logical support* of p is the set $\{q \in d(p) : w(q) = 0\}$. The logical support respects the dependency ordering in the following way:

Lemma. In a logical work, if $q \leq_d p$ in the dependency ordering, then the logical support of q is contained in the logical support of p.

Proof. If $p = q$, the lemma holds trivially. Otherwise, $q \in d(p)$, so if $r \in d(q)$ is in the logical support of q (i.e. has $w(r) = 0$), then $r \in d^2(p)$, and

[35] Russell and Whitehead, 1950.

since we have a logical work we have $r \in d(p)$, so r is in the logical support of p as well. Therefore, the logical support of p contains the logical support of q. This completes the proof.

At the turn of the century, most mathematicians operated under the assumption that the following conjecture held:

Conjecture. (Math is logical.) There exists a (possibly infinite) logical work with proposition set P such that if p is a true mathematical proposition then $p \in P$.

Up to this point, we have only concerned ourselves with the propositions and their logical dependencies. Now, we throw into the mix the citational formations contained in the proofs of these propositions.

We say a logical work is *thorough* if for all $p \in P$ we have $d(P) \subseteq \cup_{i>0} c^i(p)$. That is, one can find all propositions upon which p depends by going through the propositions cited[36] in the proof of p, going through the propositions cited in the proofs of those propositions, and so forth. This is the situation of the famous story about Hobbes's reaction to Euclid's proof of the Pythagorean theorem, where he traced each proposition from the theorem back to its underlying axioms.[37]

A logical work is *concise* if for all $p \in P$ we have the reverse containment $d(p) \supseteq \cup_{i>0} c^i(p)$. That is, no proposition is cited in the proof of a proposition on which the proposition does not depend, nor does any proposition on which the given proposition depends have any extraneous citations. We call a work *well-written* if it is both thorough and concise.

If a work is well-written, then we can completely and precisely map the dependencies of each proposition merely by following the citations.[38] This

[36]The usual form of citation is 'by Lemma such-and-such' or 'since such-and-such was shown to hold in this circumstance' and citations here are understood as explicit references to a locatable prior text, result, or argument. Certain well known theorems or results are often cited without a specific reference, as it is assumed that any appropriate standard reference would give a suitable proof or discussion. Citation also occurs in other forms which are not strictly a part of the logical structure of mathematics, as when proofs or proof arguments are explained by other proofs. For this, see Livingston, 1999, p. 872. For the role of step by step assent and its relation to mathematics in the era of digital computing, see Rotman, 2003.

[37]Shapin and Schaffer, 1985, pp. 318–319. Jesseph, 1999, p. 426.

[38]We should not forget, however, the great extent to which translation enters into

allows us to diagram the logical relationships in a work by plotting the propositions and showing the citation relations with arrows. Putting propositions of the same weight in the same row, and ordering the rows by the narrative order of the respective propositions, such diagrams can be made to simultaneously plot the narrative and logical structure of a work. Figure 5.1 shows a diagram for proposition $2 \cdot 15$ and $d(2 \cdot 15)$ from the *Principia*.[39]

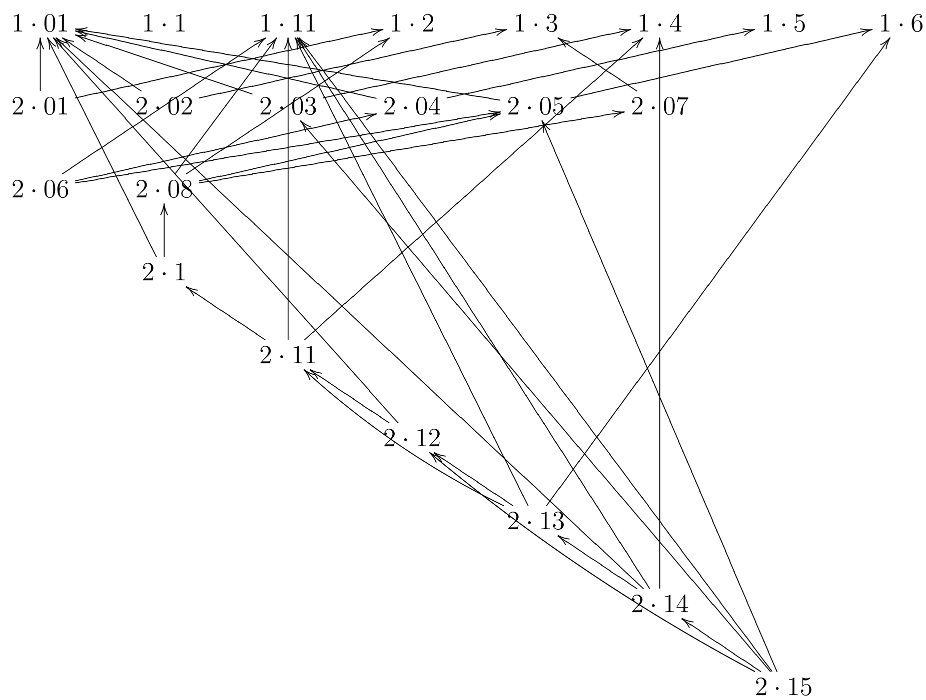

Figure 5.1: A map of internal logical citation from Russell and Whitehead's *Principia Mathematica*.

This diagram is related to acyclic dependency graphs, which arise in com-

citation-chasing. Even for Russell and Whitehead, 1950, whose work aims to completely and explicitly give all citations invoked by a proof, there remains a heavy reliance on suggestive shorthands, most of which are spelled out as glosses on the notation throughout the book.

[39]Russell and Whitehead, 1950, pp. 94–102. Proposition $1 \cdot 1$ is a special statement about deduction from true propositions, and is not explicitly cited by any of the elements of $d(2{\cdot}15)$. I should also note that Russell and Whitehead distinguish between propositions and propositional functions, a difference I have not maintained in my presentation of their work.

puter science and logic,[40] but it encodes more structural information about the narrative than do such dependency graphs. Acyclic dependency graphs are typically used to evaluate whether a system of propositions is consistent. Quite literally, if there are no cycles in these graphs, there is no circular reasoning in the corresponding propositional system. Like most analyses of mathematical production taken from within the mathematical sciences, their only concern is how things fit together logically, rather than narratively. The closest parallel to these diagrams from the established literature I have been able to locate may actually come from Bruno Latour's bibliometric studies,[41] which in turn draw from an established method of citation studies.

The central conclusion to draw is that logical narratives are both highly structured and highly complex, and depend on extensive internal citation. If one thinks of the whole body of mathematics from which an article or textbook draws as a single logically viable work, as working mathematicians frequently do, one is faced with a vast web of internal and external citations which must all hold together for weightier propositions to have any meaning or truth value. Most of this citation is sesquitextual, and so mathematicians are able to successfully defer the arduous and often practically impossible task of certifying their citational foundations at every turn. Viewed sesquitextually, citations in mathematics (not unlike citations in general) evoke supposedly necessary justificational resources and incorporate them by little more than mere mention.

A final related conclusion is that there is a necessary link between the narrative and logical orders of a mathematical work. This link is, more often than not, written off as a truism of good mathematical writing. But I maintain that good mathematical writing has an essential relationship to the rigor it attempts to produce, and that this relationship effectively delimits the field of narrative possibility within a body of mathematical work. Everything from the choice of definitional starting points and propositional benchmarks to the layout and ordering of proofs works to create more or less complicated works of mathematical proof-making which fundamentally depend on their conceptual and structural foundations.

With these structural considerations in mind, let us now return to the local linguistic production of mathematics, and the conventions which make such logical webs possible.

[40] I thank Professor Anil Nerode, Cornell University, for identifying this connection.
[41] c.f. Latour, 1987.

5.2.4 The Mathematical Sign, Revisited

The logical paradigm creates its mythic status of arbiter or pivot idiom by presenting its underpinnings as an infallible code.[42] In contrast to the interpretation of psychic imagery, where no such code exists, logical representation requires a radical separation between the signifier and signified,[43] barring rational argument from crossing between formal expressions and that which they would claim to represent. Conventional, or informal, mathematics, including nearly all mathematical work that is published and accorded truth value, makes use of both narrative description and logical expressions and quantifiers in presenting proofs.[44] Narrative description, as mathematicians are plainly aware, has substantial shortcomings as a stable representational system.[45] Logical proofs attempt to short circuit this shortcoming by eliminating conventional narratives from the process of proving. Instead, proofs become a sequence of logical propositions or sentences, with one sentence obtained from prior sentences by the application of an appropriately cited transformation from Peano's, or another's, logical calculus. Symbols in mathematics become as rigidly policed as pollution or other taboos and sources of contagion from sociological investigations such as those of Mary Douglas.[46]

The way mathematical concepts are built makes all names proper, to the extent that they aspire to be outside of any fixed idiom or vernacular, but rather to be fixed to a single immutable class of concept-objects. Words which are perfectly intelligible outside of mathematics are attached to highly specific notions, transforming the words into signs for specific concepts with elaborate conceptual lineages. In the above discussion of the narrative and logical structure of Russell's logical calculus, terms as general as *concise*, *thorough*, and *well-written* were given precise set and function-theoretic meanings. Even if particular words from mathematics are translated between common languages,[47] there is a sense in which the concept is never really translated, or at least supposed not to be translated. Proper names are never strictly a part of a particular language, so making names proper

[42]Here, I use *code* in the sense of idiom or linguistic system, and not in the sense of Rotman's Code from chapter 4.

[43]Derrida, 1978, p. 209.

[44]c.f. MacKenzie, 2004, p. 75.

[45]Lakatos, 1979, pp. 51–53.

[46]c.f. Bloor, 1978, p. 254.

[47]For example, open neighborhoods become *voisinages ouverts* in Dieudonné, 1944.

liberates them from linguistic parochialism.[48] The logical paradigm would remove even recourse to any spoken language, for while logical symbols have corresponding verbal expressions, their transmission and rules of operation happen entirely on the page or the chalkboard. Every word in logic is a proper name, above the constraints of narrative description.

Logical symbols thereby acquire a use-meaning remarkably similar to the meaning described by Wittgenstein.[49] Curiously, as will be elaborated below, logic is a language made almost purely of signifiers, and which effaces the signified in its attempt to universalize it. These signifiers, however, remain active elements in logical manipulation, and are no mere dead letters. Like maps, these signifiers become metasigns.[50] All the while, logical signs both promise and mask the ideal mathematical objects they purport to represent like Benjamin's figure of the royal robe in "The Task of the Translator."[51] Hidden is the essential instability of the well-cloaked ideal mathematical objects.[52]

5.2.5 Problems of Definition

Short of lexicography, no discipline defines its terms with such ardor as mathematics, and no mathematician is as concerned with definitions as a logician. Any system of symbols must deal with the possibility of multiple definitions and understandings of operative terminology.[53] There will always be competing conceptual systems and alternative terminologies, and behind each system is a different locus of power and authority.[54] In informal mathematics, these terminological struggles are pronounced, and often quite vicious. Definitions are dismissed as inadequate, or as permitting teratological examples to be accorded the status of legitimate mathematical objects.[55]

Russell deals with the multiplicity of significations by introducing the notion of a logical constant. His frequent example is the proposition

[48] Derrida, 2002, p. 109.
[49] Wittgenstein, 1956, IV: 5.
[50] Thongchai, 1997, p. 138.
[51] Benjamin, 1968, p. 75. Derrida, 2002, p. 125.
[52] Lakatos, 1979, pp. 136–139.
[53] Thongchai, 1997, p. 171.
[54] Ibid., p. 59.
[55] c.f. Dieudonné, 1944. Lakatos, 1979, is filled with examples of competing rhetorics of teratology.

If A is a man then A is mortal.

Here, A stands as a single abstract instance in place of a large group of possible mathematical objects, and so denotes a *class*.[56] It should be noted that Russell also takes care to state what he means by denotation, which refers to a "term connected in a certain peculiar way with the concept,"[57] although his definition does not meet the standard of analytic precision he would require of propositions of a more strictly mathematical nature. That he defines denotation at all shows a dogged commitment to explicitness which his parsing of the term as a "certain peculiar" relation betrays as mostly ornamental. Indeed, relations between symbols are often described as peculiar, or queer.[58] Especially when one tries to pin it down, the vagaries of linguistic articulation make language a remarkably unspecifiable and undefinable medium for symbolic production.

If the relationship between signifier and signified is left ambiguous, the implications for the use of abstract signifiers is treated with excessive rigor. Russell begins his 1903 *Principles of Mathematics* by defining pure mathematics as the class of all propositions containing only logical constants.[59] Since the status of logical constants is to be a contested domain, Russell must make their intent and use explicit from the outset. Immediately after proposing his definition of pure mathematics, Russell takes pains to point out that it is a definition capable of exact justification.[60] This claim is dubious at best, but stating it at the beginning of the book does two things. First, it gives the claim the status of something pre-given, which doesn't need to be logically developed by a lengthy exposition. Second, it makes accepting the claim, at least provisionally, a prerequisite for continuing to read the book.[61]

One cannot forget the material situation that Russell's (and later, Russell and Whitehead's) is a long book with many dense and complicated claims, all with varying degrees of justification. One of the most touted strengths of mathematics is that it starts with clear definitions and proceeds by clear logic

[56] Russell, 1937, p. 8.

[57] Ibid., p. 56.

[58] See Wittgenstein, 1956, I: 127.

[59] Russell, 1937, p. 1.

[60] Ibid., p. 2.

[61] It cannot here escape notice that most mathematical theories are not based on a uniformly agreed-upon set of fundamental and explicitly articulated axioms, although all follow from some sort of starting point rooted in assumed familiarity with certain background concepts and a set of formal definitions.

to an explicit solution.[62] Definitions, then, delineate the terms and methods of an argument in mathematics, and create the space, possibility, and staging ground for mathematical disputation. If Russell's understanding of logic later seems inadequate to the problems he attempts to solve, a defender of Russell would need only to refer back to the outset of the volume for a neatly segmented account of the rules of engagement, in the form of organizing definitions, which can be carefully chosen to lend plausibility to whatever argument is being advanced by the author.[63] If mathematics is to start with its definitions at the beginning, it is for reasons beyond mere convenience or propriety.

There is another conclusion to be drawn from the length and complexity of Russell's, and particularly Russell and Whitehead's, early twentieth century works.[64] One can read either work as a *reductio ad absurdum* of the program of explicit logical and definitional justification for mathematics. While many have read or attempted to read the *Principia Mathematica*, it would be impossible to read it as Russell and Whitehead intend. One cannot possibly imagine Hobbes, upon reading that $1 + 1 = 2$, which is given as a "purely formal" definition more than half way through the three volume work,[65] or any other statement of mathematical importance to the authors, and tracing the statement or proposition back through hundreds upon hundreds of pages of dense logical symbolic expressions. One certainly couldn't imagine such a tracing if one required that it illuminate the original proposition and produce a meaningful understanding and comprehension of its truth value. These works were not written to produce truths in the way they claim to, by developing comprehension and securing assent in and from the reader in a purely logical and grounded fashion. Something else is at play which gives them the proof-making power which they claim to possess.

Assent, were it is achieved, is made through the production of local citational orders, where each step of the proof is in a certain realizable sense checkable, but where the entire network of results undergirding all but the most elementary of propositions is completely unsurveyable and largely incomprehensible as a whole. The structural and logical unity of the whole is built and maintained by managing the micro-relations between logical or

[62] Shapin and Schaffer, 1985, p. 100.

[63] The presentation of definitions at the beginning elides the extent to which their development is proof-driven. Lakatos, 1979. Livingston, 1999, p. 879.

[64] Russell, 1937. Russell and Whitehead, 1950.

[65] Russell and Whitehead, 1950, Vol. II, p. 467.

analytic expressions in the immediate context of their articulation and deployment. The global hypothesis of a logically sound system, on which the local hypothesis of a valid proposition depends, is one which can ultimately only be confirmed through a projection of the local relation of citational vigilance which certifies that each part functions as it should within the putative whole.[66] Once again, the relationship between the narrative and citational ordering of the work becomes instrumental, as the former provides a tangible guarantee on the soundness of the latter.

A final dimension of Russell's opening comes to light in consideration of what he did not write. The vast majority of his expository material discusses the status of logical constants and propositions as a whole. Left between the lines is the status of the logical terms connecting the statements, including mathematical prepositions and logical quantifiers such as 'there exists' ($\exists$) and 'for all' ($\forall$). These terms, too, can be contested domains, but they must be absolutely rigid in order for a logical argument to even begin.[67] By assuming their unimpeachable applicability, Russell removes logical terms not only from the sphere of contention, but from the entire realm of discussion.

5.2.6 Metaphor and Mimesis

At the heart of the problem of definition in mathematics is the role and function of metaphor. Metaphors operate on many levels in the mathematical sign system. The rules of symbolic logic function as metaphoric substitutes for supposedly basic human logical intellect. Any commerce in ideal objects is bound to use metaphoric devices to represent and break apart details and structures of inaccessible generalities. Most proofs, and especially geometric proofs, make extensive use of synecdoche in treating general classes of related mathematical objects. Finally, because math must claim access to infinite classes of objects, all proofs and operations must be defined through the use of metaphor, where corresponding operations on related objects are performed *in the same manner* as those for a few sample objects.

[66]Governmentality, it would seem, is not limited to state control of people. See Foucault, 1991.

[67]Bloor, 1978, p. 267–268. Logicians in the latter part of the twentieth and turn of the twenty-first centuries have, in fact, explored what can be made of logical calculuses with different varieties of conjunctions and prepositions, and even different forms of truth valuation. For a thought-provoking analysis of the articulation of the latter, see Rosental, 2004.

Metaphors are indispensable in mathematical imagery because of the vast expanse of ideal notions claimed by the study of mathematics. The first limitation any mathematician encounters is how to discuss objects which can never be drawn, or even imagined. The same proof of the Pythagorean theorem which works for a triangle of side lengths respectively 3, 4, and 5 inches must work for a related triangle on the scale of miles, as well as one with a base nanometers across and taller than the diameter of the universe. Even notions like width and height are complicated on this scale, as we will see below. Metaphors and their technologies of representation must remain closely interlinked to have any hope of methodological efficacy.[68]

Calculation is one sphere where this is immediately apparent. When one learns to multiply, one rarely deals with numbers of more than a few digits. The rules of calculation are necessarily ambiguous in order to allow simple model computations to explain how to do computations whose time required could in practice exceed the age of the universe if ever carried out rigorously with the same methods as for the small examples.[69] With such metaphoric extension, there is always the risk of misapplication. One can easily wonder if everyone has been computing 12×12 incorrectly all along.[70] It is a computation carried out daily in elementary schools across the world, as well as in groceries and bakeries and a whole range of other professional and informal settings. Results differ, if not in number, then in what is being counted, what its status is as a counted object, and what the operation in 12×12 means or represents. For Wittgenstein, math's role as a practical tool, independent of its truth-value, makes the answer to the preceding question irrelevant, but this inquiry has fundamental consequences for any pure mathematics claiming to encompass all calculation.

Other problems can arise when examples take the place of ideal objects in mathematical discourse. Examples are almost never proof-neutral, and the particular examples used can have drastic effects on the course, applicability, and reception of a proof.[71] The Pythagoreans offer a well-mythologized case study in the reception of a mathematical proof in context of a dominant set of examples.[72] Their model of measurement involved counting standard unit lengths comprising the total length to be measured. This model articulated

[68]Derrida, 1978, p. 228.
[69]Wittgenstein, 1956, I: 113.
[70]Ibid., I: 134.
[71]Livingston, 1986, pp. 39–40. This problem will be explored in depth in chapter 6.
[72]Wilder, 1973, pp. 88–90.

a basic notion and assumption of commensurability which required the existence of common units by which any pair of lengths could be mutually measured. It is well known that the Pythagoreans' model of measurement could not be reconciled with the existence of irrational numbers, and a proof of the irrationality of the square root of two had stark affects on their mathematical community. In general, because ideas and metaphors come from experience, it is impossible to excise them from one's limited range of comprehension and pervasive social conditioning, and one thus always risks employing mutually incommensurable or contradictory models in one's mathematics.[73]

It would be one thing if mathematicians and logicians were content with metaphor as a model of inaccessible mathematical figures. In order for math to have the foundational relevance implied by pure language, however, objects of mathematical practice must be raised from the status of metaphoric to mimetic representations. Not only must a triangle in a geometric proof take the logical place of an arbitrary triangle (whatever that means, and its meaning is by no means stable), but it must *perform* in every way like the abstract Platonic ideal of the Triangle.[74] This is true not only of triangles in geometric diagrams, but also of algebraic analogues to such triangles, which were made to perform by Cauchy in a properly mimetic fashion by means of a range of regulating practices. Figurative symbols must be represented as mimetic in order to gain the status of their ideal brethren, and this accords to them a great significance.[75] This mimetic function is always deliberate

[73]Bloor, 1978, p. 268.

[74]See Livingston, 1999, p. 873 on the representational status of objects in material proofs. There are two parts to his argument. First, Livingston claims that symbols or figures drawn in mathematical proofs are "not really" the ideal mathematical objects they purport to represent. The character '1' is not the number 1, the triangle in a Euclidean proof diagram is not really a Euclidean triangle, and so forth. Second, it suffices that these representations be merely "adequate representations" and they need not be complete substitutes for what they claim to represent. For the latter, see Ibid., p. 873 n. 14. The material matters of mathematics are distinguished from its transcendental subject matter. It is possible that Livingston underplays here the core of materialism in his argument in order to sidestep debates over mathematical Platonism. See Ibid., p. 876 for his materialist claims. Whatever his motives, he is correct in pointing out that, whatever the status of the material figures of mathematics is, such figures are always *understood* to represent far more than what is actually materially incarnate. Netz, 1999, p. 242 explains that Euclidean triangles were then understood only as particular occurrences, which achieved generality through a concrete surveying and recombining instead of through any *a priori* claim to generality from the method.

[75]Thongchai, 1997, p. 53.

and productive, and is constitutive of the ideal subject-figure-sign itself.[76]

Symbolic mimesis involves a special and widely discussed form of subjectivity. Mimesis always threatens the pluralization and fragmentation of a subject.[77] If the mimetic object is a symbol, with the function of signification, this both breathes life into the symbol and, in a sense, takes life from the target of mimesis. One literally observes the sort of life-death confrontation of the fragmented self found in Hegel's description of lordship and bondage in the constitution of the subject.[78] This figuration can be seen as an instance of Lacan's subjective formation in the face of a mirror.[79] The mirror gives a good model for mimetic subjectivity, and a more detailed analysis of its role in mathematics, where a few lines on a page 'mirror' impossibly complex and abstract figures, is certainly warranted.

The mirror is an apt model for speculation and spectrality, both of which shed light on the structure of mimesis. In "To Speculate—On 'Freud,' " Derrida lists over the course of one paragraph at least five distinct senses of the word speculation.[80] They are, roughly, specular reflection, the production of surplus value, that which is given, a mode of research, and the operation of Freud's writing. Beyond these, notions of the spectacle, introspection, retrospection, inspection, and so forth can all be said to share the specular structure of a mirror. The economic element of specularity makes an explicit appearance in discussions of proof-making in terms of currency and exchange.[81] Proofs wed speculation as a mode of research to speculation as a mode of exchange by creating symbolic fora where hypotheses from the former mode are deployed alongside mathematical formulations and manipulations which are transferred and evaluated as in the latter mode. The proof, moreover, is itself a spectacle of mimed production.[82]

Mirrors have a somewhat unconventional function in the structure of mathematical mimesis. In mathematics, the mirror image, at best an imperfect reproduction, is the reflection of, depending on how it is conceived, an infinite multiplicity of forms or something with no form at all. The math-

[76] Lacoue-Labarthe, 1989, p. 85.
[77] Ibid., p. 129.
[78] Hegel, 1977, p. 187.
[79] Lacoue-Labarthe, 1989, p. 127.
[80] Derrida, 1987, pp. 283–284.
[81] c.f. Wittgenstein, 1956, I: 152.
[82] A wide body of Latour's work elaborates on the spectacle of scientific production. In particular, Latour and Woolgar, 1986, and Latour, 1987. See also Bourdieu, 1999.

ematical mirror is thus a productive one. The image in the mirror is jarring once one realizes that it is not, in fact, the reflection of any one *thing*. The status of the ideal object as a thing must then be constructed through the figure of mimesis and the purported accuracy of representation and exemplarity in order to dispel this horrifying possibility. The reflected is more than simply subsumed by the reflection. It is already materialized only by way of the reflection. In mathematical proof, the reflection becomes the prior subject.

5.2.7 Ruptured Ideality

We have now discussed several of the ways in which the linguistic position of mathematics imagines itself and generates discourses. Before exploring in more detail the ways in which these phenomena manifest themselves in practice, it would be valuable to recall some of the methodological stakes of the structure of the logical paradigm. Logic construes itself as a pure language capable of mediating and producing truth. For Husserl, the normative autonomy of this system would be central in order to begin an account of subjectivity in general.[83] It was, after all, the failure of logical ideality to produce a fruitful closure which made necessary the phenomenological attitude.[84] But structural phenomenology also fails on this front, because this same inaccessibility of closure makes any possible essences of consciousness inexact, and hence inaccessible themselves.[85]

Platonism has definite epistemic ties to an old guard of ontological figuration which effaces the production of the subject.[86] It is no coincidence that the style of mathematical proof-analysis described by Lakatos emerged only after a series of Prussian university reforms in the early 19th century.[87] Mathematical methods and theory bear resemblances to the broader onto-philosophical climates in which they were produced and which they had a role in creating. We saw in chapter 3 how integrally connected Cauchy's politics and circumstances were to his mathematics. A consequence is that because there are no stable social essences, there can be no stable platonic

[83]Derrida, 1978, p. 158.
[84]Ibid., p. 159.
[85]Ibid., p. 162.
[86]Lacoue-Labarthe, 1989, p. 71.
[87]Bloor, 1978, pp. 263–264.

essences or ideal forms in mathematics.[88] Essential commonality between systems is itself a fabricated notion which excludes an always present and always haunting (indeed spectral) potential for propositions to be true in one system and false in another, and for there to be no decidability in and between competing systems.[89] The same metalanguage which is impossible for narrative discourse is impossible for logic as well.[90]

5.3 Math as a Practice

I shall now turn to the lived and practiced elements of mathematics as a linguistic system. As such, mathematics participates in a communicative order which fundamentally shapes its social space of production. This communicative order is one laden with relations of position, status, and meaning. As a language, mathematics enacts a framework for relating concepts which simultaneously produces an embodied framework for relating people.

5.3.1 Address

In the production of a proof or the assertion of a logical proposition, it is easy to overlook the circumstances of creation and address which make the proof or proposition possible.[91] Indeed, a central proposition behind the purity of logic is that its truth is utterly independent of the time and place in which it is conceived and enunciated. If mathematical objects and mathematical truths pre-exist their articulation in a Platonic realm of ideal truth, then they have no need for exposition in order to exist, and so there is a vested interest for the logical paradigm in the suppression of the actual acts of generation and transmission of mathematics. If, on the other hand, mathematical objects are the contingent fabrications of human whimsy, and exist only insofar as they are believed to be ideal, then mathematics as a practice is without foundation. It is scary to think that mathematics could be the pure creation of practice, and yet this admittedly extreme conclusion has become more and more plausible as a general sociology of knowledge becomes less and less

[88] Ibid., p. 257.

[89] Wittgenstein, 1956, Appendix I: 8.

[90] Derrida, 1998, p. 22. Both metadiscourses are impossible insofar as both underlying systems always risk incommensurability, undecidability, and absurdity.

[91] Wittgenstein, 1956, Appendix I: 6.

guarded about investigating the fundamental underpinnings of mathematical practice.[92]

Mathematical rigor is, more than anything else, a social technology, and one which is made and maintained by the constant narrative production and enforcement of millions of practitioners at many levels throughout the world.[93] Its very grammar of enunciation fabricates a sense of disembodiment which makes it possible to believe there would be math without mathematicians, or even without humans. Ontological formulations turn narratives into propositions.[94] The speaker is effaced through a pronominal disjunction, often through a shifting use of the first person plural,[95] and this creates a space within which enunciation can be imagined without a speaker.[96] The conceit that "...a proof must be capable of being reproduced by mere copying" further removes the speaker from the mathematics.[97] While the role of individual provers is acknowledged, for instance in the system of theorem nomenclature,[98] the proof itself must be immune from the vagaries of speech.

Ignored even more than the speaker of a mathematical text is the audience. Often, textbooks will give the reader token mention in an introduction or preface, and frequently there is a statement that the reader will have a "certain mathematical maturity," which is to say that the reader has been socialized to accept the Platonic order of the production of mathematical truth embraced by the logical paradigm.[99] The mathematical speaker can be said to have confused address with communication in presenting a proof as though its truth was pre-given and independent of the audience's comprehension or assent. This is a hallmark of the regime of homolingual address.[100]

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[92]Bloor, 1976 and 1978.

[93]MacKenzie, 2001, p. 12. Insofar as truth is something in which one believes, we might apply Derrida's formulation of belief as something invoked in each and every relationship with the other. Derrida, 2005, p. 77.

[94]Wittgenstein, 1956, Appendix I: 4.

[95]I am certainly guilty of this disjunction in section 5.2.3 above.

[96]Sakai, 1997, pp. 12, 47.

[97]Wittgenstein, 1956, III: 41.

[98]Bourdieu, 1999, includes a discussion of the grand social function of eponymy in the sciences, a function which is arguably more pronounced in a discipline such as mathematics where one is unlikely to garner a patent or copyright for many instances of one's original work. Publication, however, does not necessarily correlate to reputation in mathematics. See Jesseph, 1999, p. 429.

[99]Lakatos, 1979, p. 142.

[100]Traces, 2006, p. 7. The term is defined in detail in Sakai, 1997, pp. 3–4.

Thus, when a reader does not subscribe to the conclusion of a proof it is written off as either having been the result of a poorly worded proof or of a reliance by the proof on concepts beyond the experience of the reader, and is never understood as non-comprehension of the proof as such,[101] nor is it understood to judge the ultimate comprehensibility of the argument. Within mathematical proof-making, there is no room for varying degrees of proving. Either a result is proven or it isn't. Unlike heterolingual address, it cannot admit partial proof or comprehension, at least within the logical paradigm.[102]

A curious phenomenon is that because, in the logical paradigm, math is posited as an external ideal, there is under this paradigm no original speaker or author in the strict sense of the idea as it would apply to other creative works. Mathematical texts, rather, have the character of the perfectly translatable sacred text posited by Benjamin. Mathematical writers then become translators from the outset. The position of the translator is itself a highly contested and significant domain.[103] Typically, the relationship of address mediated by translation can be modeled as an exchange between the three figures of addresser, translator, and addressee.[104] We saw a similar formation in the observer model's brand of testimony.[105] Things become more complicated when the addresser not only must be at best implicit in the address of the translator, but cannot have made an original address at all. The translator always must promise on behalf of someone else, but in math the translator promises on behalf of someone else who is no one else.[106] Identity is abstracted from communication.[107] Consequently, the positing of the mathematical speaker as a sort of originary translator forms the basis of a system of identification which is inextricably complicit with the logical paradigm of address.

[101] Sakai, 1997, p. 6.
[102] Ibid., p. 4.
[103] Ibid., p. 5.
[104] Ibid., p. 11.
[105] See chapter 2, as well as the model's application to mathematical semiosis in chapter 4.
[106] Ibid., p. 11.
[107] Ibid., p. 10.

5.3.2 Time and Practice

One can think of the methods of formal mathematics as a collection of time saving techniques. The logical paradigm claims to deal with properties of temporal priority, existing outside of the time of lived experience.[108] Logic, however, is practiced within the realm of lived experience, and in varying times and places. For the logician, this contradiction is reconciled by recourse to the priority of Platonic ideals and pure language. But the Platonic is always mediated by translation, and translation is a ubiquitous social activity in time and space, so even logicians must consider the temporality of their work.[109] As a practice, math operates by repeating, or at least claiming the possibility of repetition. Repetition produces witnesses and lends surety to mathematical claims.[110] Repetition also opens the space of translation, dividing and recombining transgressive instantiations of a pure idiom.[111] Here, repetition refers not only to the repeated use of the signs and conventions of the logical idiom, as it does for Derrida, but also the repeated experience of and encounter with the act of proving and knowledge iteration.

Proofs are not the only forms of mathematical practice put outside of time. Calculation is de-contextualized in several ways. A question such as 'when does 2+2 = 4?' is not intelligible in the logical paradigm. Nonetheless, there are many modes of counting where $2 + 2$ does not give 4.[112] There are even times and contexts where the categories denoted by '2,' '4,' or even '+' and '=' are not coherent. Wittgenstein argues that

> By accepting a proposition as a matter of course, we also release it from all responsibility in face of experience.
> In the course of the proof our way of seeing is changed—and it does not detract from this that it is connected with experience.[113]

The face of experience is often a grim one. In a practical sense, big calculations or proofs of theorems with more than a fairly elementary struc-

[108]Russell, 1937, p. 12. For the importance of the passage and marking of time to modernity, see Latour, 1993, p. 10.
[109]Traces, 2006, p. 9. For another sense in which translation is social, see Callon, 1999, p. 81.
[110]Shapin and Schaffer, 1985, pp. 60–61.
[111]Derrida, 1978, p. 213.
[112]Wittgenstein, 1956, I: 37–38.
[113]Wittgenstein, 1956, III: 30.

ture cannot be performed in the calculus of symbolic logic.[114] Even when they can be, such supposedly infallible logical deductions are subject to a *calculation fatigue*, which renders their outcomes unreliable, at least when compared to the ideal sense to which they might aspire.[115] Instead, short-cuts are taken, such as using small measurements or computations as models for large ones.[116] We may not be able to measure the distance from the Earth to the Sun by stringing together meter sticks, but if we could, we would get the same result as with a more practical method of measurement. Another way to combat calculation fatigue is through the use of surveyable notation, which can be easily scanned for correctness by a time-pressed logician.[117]

Logic's atemporality is constructed in large part through the structure of mathematical narratives. The archetypal mathematical paper begins with a short expository motivation, states definitions, gives one or more examples or constructions, states a theorem, states and proves several supporting lemmas, proves the main theorem, and closes with some applications or open problems. This representation of the proof completely obscures its production.[118] All reporting and discussion of mathematical results begins with the result in hand, and more often than not will pretend as though the result has always existed in its present form, save for a possible allusion to the novelty of the current proof.[119] By telling a narrative more in line with the logical paradigm of orderly discovery of mathematical truth than with the actual heuristic principles involved in the production of the proof, a mathematician or logician projects the former heuristic in place of the latter. A rupture in mathematical thought is transformed into continuity by the narrative re-imagining of history.[120] Narrativization and the account of a proof are figured as exceptional acts in the metaphysical calculus of proof-making.[121]

Structure and systematicity depend on the synchronic unity of the system in question. Just as the *langue* is a regulative idea which makes systematic linguistics possible, the image of an ideal space of mathematics makes sys-

[114]Ibid., II: 3.
[115]Ibid., II: 17.
[116]Ibid., II: 4.
[117]Ibid., II: 16.
[118]Lakatos, 1979, pp. 142–143. Schuster, 1980, p. 58 gives an example of Descartes's method-story failing to line up with a reconstruction of its development.
[119]Livingston, 1986, p. x.
[120]Thongchai, 1997, pp. 149–150.
[121]Sakai, 1997, p. 3.

tematic logical exposition possible.[122] If $2 + 2$ cannot always be replaced by 4 and vice versa, then the mathematician in the joke cannot just leave the bucket of water on the floor, and logicians can never simply build on the work and labor of previous logicians. That is, they cannot ignore their own work and labor, as well as that of their predecessors and contemporaries, by taking mathematical results outside of time and context and citing them as solved problems or applicable and immutable truths. If math is context-sensitive then it is also impossible, or rather, it is impossible as it is imagined in the logical paradigm. Mathematicians must, therefore, tell a story of mathematics ripped from its context—a context which can only return when a mathematician needs to retrospectively explain away a misunderstanding or a false theorem. Context protects mathematics as a practice insofar as it can be hidden away until it is needed.

5.3.3 Translation and Practice

> ... translation is another name for the impossible.[123]

Everything and nothing, according to Derrida, is translatable. That is, everything submits itself to translation, and nothing can be accurately transmitted from one idiom to another. Indeed, accuracy itself is a contestable category within different conceptual systems and idioms.[124] Informal math can be seen as subject to the rules of logic only insofar as it can be translated or recoded into symbolic logic's propositional calculus, and this coding is by no means natural or perfectly rigorous.[125] There would seem to be at least two languages in mathematics: the language of the world, of applications, and of vague intuitional notions, and the language of truth, of logic, and of decidability. On the one hand, logic must rigidly separate itself from informal mathematics in order to protect its access to ideal truth. On the other hand, logic must create a means by which informal mathematics can be understood in logical terms, in the manner of Russell's texts. This is not the only case where the practice of translation is the very thing which institutes and articulates linguistic difference.[126] Logic has the power to certify proofs

[122] Ibid., p. 57.
[123] Derrida, 1998, p. 57
[124] MacKenzie, Accuracy, 1999.
[125] Bloor, 1978, p. 269.
[126] Sakai, 1997, p. 2.

because it is said to be absolutely separate from common mathematics, and because it is said to be absolutely translatable to and from the same.

The question of why people believe proofs has received remarkably little attention.[127] Within the confines of the present discussion, a theory of the role of the logical paradigm could shed light on the complexities of proof acceptance, given logic's status as a mediating language. Indeed, proofs are often best characterized by their logical compulsion.[128] This logic is carried within a specific deployment of linguistic conventions, regardless of whether or not the proof is strictly one of symbolic logic.[129]

Russell points out that, drawing from the Euclidean model of proof and knowledge production, proofs have both material and formal implications.[130] It is the material implication of a proof which is convincing for Wittgenstein. Geometric application is a principal staging ground upon which proofs can, so to speak, prove themselves.[131] In fact, for Wittgenstein, there is nothing to a proof beyond its corresponding geometrical certainty, which would make Russell's system a moot point, and would render Cauchy's system of transmuting geometric certainty into algebra a purely redundant enterprise.[132] In order for the logical system to function, the formal implications of a proof have to take precedence, for those are the implications corresponding to ideal forms which can then be used in later proofs about ever more complex idealities. Of course, for such a production to be valid, one must make the impossible guarantee that one has not overlooked anything.[133]

Logical proofs operate by mapping intuitive concepts onto corresponding symbols, manipulating the symbols, and then returning the new symbols back to their original conceptual idiom. Consequently, acceptance of a proof depends on *both* the acceptance of the logical system and its manipulations *and* the idea that the logic does in fact correspond to the intended object of the proof.[134] For this reason, Wittgenstein defines logic as the "superficial

[127] Bloor, Lakatos, Livingston, and MacKenzie's work notwithstanding.

[128] Wittgenstein, 1956, I: 117.

[129] Ibid., I: 152.

[130] Russell, 1937, p. 15.

[131] Wittgenstein, 1956, II: 38.

[132] Ibid., II: 43.

[133] Ibid., II: 86. Wittgenstein's argument here is not that proofs themselves must always overlook something, but rather that they can never fully account for the field over which they look, i.e. their domain of applicability. Chapter 3 points to the crucial role of such domains in assessing the overall truth value and rigor of a proof.

[134] Ibid., II: 53.

interpretation of forms of our everyday language as an analysis of structures of facts."[135] Part of the required translation involves a specialized vision, where, for instance, a logician presented with objects A and B will see '$A \vee B$.'[136]

Translating common or vague notions into logic presents two additional difficulties. First, logic deals with highly abstracted objects as logical constants, and so falsely treats mathematics as a content-less web of inferences.[137] Second, the extent to which logical constants can capture the essential properties of a notion from informal mathematics is severely limited by different standards and values in definition-making. Many details of vague concepts are likely to be lost in the translation to an abstract symbolic medium, and these lost details or unrealized intuitions are then easily dismissed or forgotten when regarded again through the lens of logical abstraction.[138] Russell's opening declarations in *The Principles of Mathematics* might then be restated, as he did sixty years later, as follows: "mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." [139]

5.3.4 The Multiplicity of Mathematics

It is clear that not all mathematics can have unfettered access to truth, so for the logical paradigm to claim such access it must rigorously separate its mode of practice from competing modes of mathematical production. Its principal means of doing so is through translation, which sets up the system of symbolic logic as separate from other means of mathematical proof. This move toward the independence of logic then retroactively creates other systems of production and understanding through the mediation of translation.[140] Since meta-translation is not possible, the possibility of translating a mathematical idea into symbolic logic and the rules for such a translation must also, therefore, be posited retrospectively.[141] When Russell writes about the rules of

[135] Ibid., IV: 48.
[136] Ibid., V: 47.
[137] Bloor, 1978, p. 268. For Russell and Whitehead, 1950, this contentless property of logical mathematics is one of its greatest strengths.
[138] Lakatos, 1979, p. 122.
[139] In Bloor, 1978, p. 267.
[140] Traces, 2006, p. 20.
[141] Sakai, 1997, p. 54.

logical production, he does so from within a system where logic has already been posited as a separate idiomatic unity through the translations of those such as Peano.[142]

One only understands the work of logic by imagining the corresponding operations in intuitive arithmetic or geometric systems. Only through annexing these concepts by setting up symbolic equivalences can symbolic logic claim a coherent identity as a system.[143] There is thus always a sort of bilingualism harbored in logical practice, where the mother tongue of informal mathematics (which, it should be noted, is nobody's mother tongue in the conventional sense, and is posited as everybody's mother tongue in the schema of universal logic) becomes part of a heterogeneous web of understanding mediated by a foreign pivot. Informal mathematics doesn't 'speak' in logic, but rather translates into it, thereby positing itself as a separate linguistic entity.[144] Logic objectifies these mother-concepts, which in turn makes disciplinary loyalty possible through loyalty to corresponding symbolic idioms.[145]

When math is practiced heterogeneously as a mix of informal intuition and symbolic logic, linguistic loyalties and dispositions can be noticeably complicated.[146] It is thus rare to find logicians doing extensive informal mathematics, or vice versa. Where such mixing does occur, as it does in algorithmic fields such as computer science and operations research, systems of reporting and journal organization provide an extra layer of insulation protecting logical separatism. Yet logic would pervade all of mathematics as an organizing principle. One can approach this along the lines of the relationship between a mother tongue and its dialects. Here, the mother tongue would be the entire system of mathematics as imagined by the logical paradigm, and the dialects would be the individual modes of mathematical practice. On the one hand, the mother tongue cannot be imagined without the dialects, but on the other the dialects don't exist as dialects without the organizing figure of the mother tongue.[147]

We close this chapter with a question: What does it mean for mathematics to be rooted in homolingual address, and what would a math-as-

[142] See section 5.2.3.
[143] Berman, 1992, p. 32.
[144] Ibid., p. 148.
[145] Thongchai, 1997, p. 133.
[146] Sakai, 1997, Chapter 1, especially p. 37.
[147] Berman, 1992, p. 166.

heterolingual-address look like?

As it has been presented here, mathematics can readily be imagined without the regulating influence of homolingual address. One avenue for such an imagination would be the practice-rooted articulations of the discipline put forward by Wittgenstein. Such frameworks have the advantage of admitting different modes of comprehension in mathematical practice. Moreover, they recognizes the crucial difference between address and communication. Its relationship to ideality can be summed up as "mathematics is *normative*. But 'norm' does not mean the same thing as 'ideal'."[148] Math can be applied, in a loose sense of the word, without being pure in the sense of logic.[149] Mathematics-as-practice acknowledges the grammaticality of mathematics, the language of production, and the hidden contingencies in every address.[150] Rather than one towering system, mathematics is a surprisingly *ad hoc* collection of techniques of proof, understanding, and intuition building.[151] Mathematics would give up its impossible aspiration to be a language without a people, and instead become a practice with a purpose.

[148]Wittgenstein, 1956, V: 40.
[149]Ibid., III: 15.
[150]Ibid., II: 26.
[151]Ibid., II: 46.

Chapter 6

Witnessing in Mathematics: A Social Model

It is one of the great ironies of scientific discourse that the method taken to be the most unimpeachable guarantor of epistemic surety also offers one of the most fecund examples of the social organization and production of knowledge. The mathematical proof has long been a contested domain for those, such as logicians or analysts of in the tradition of Cauchy, who seek to increase its rigor, and, in rare cases such as that of Imre Lakatos,[1] who seek to expose the many contingencies at work in a formal proof. Nonetheless, the mathematical proof as a means of creating certain knowledge has, at least in its ideal form, a mythic status which is almost invariably viewed as unaffected by its social means of production.

This chapter will take as its model Eric Livingston's introduction to his *Ethnomethodological Foundations of Mathematics*.[2] I shall offer two elementary proofs from Euclidean geometry, the gold standard of mathematical proof-making, and then discuss the many ways in which the proofs rest on a social, rather than a transcendent epistemic foundation. To close, I shall examine Cauchy's two proofs of the Intermediate Value Theorem from within the analytic framework developed in the chapter. In addition to Livingston's particular ethnomethodological concerns, I shall uncover questions of professional vision, knowledge communities, archetyping, externalization and internalization, abstraction, socialization, storyability, historicity, and

[1] c.f. Lakatos, 1979.

[2] Livingston, 1986, pp. 1–15.

metaphoricity, all of which form indispensable social tools towards the production and validation of mathematical proofs.

While arguments in this chapter will certainly draw on semiotic and linguistic issues, the focus will be on the particular ways in which such conditions enunciate and define a community of mathematical practitioners, and on the ways in which such communities create semiotic or linguistic norms and practices. My concern will not be so much with the putative or structural meaning of a particular representational practice as with its social meaning. For it is the social meaning of a practice which renders it sharable and workable. Any attempt to understand how mathematics *works* must necessarily attend to how mathematics is *done*, which is to say how mathematics exists in, around, of, by, and for people.

6.1 Two Propositions from Euclidean Geometry

6.1.1 A Note on Choosing Propositions

One of the most important concepts in any social understanding of an intellectual system is that of exemplarity. The way a community or practitioner chooses examples by and through which to make an argument is of crucial importance to the argumentative and conceptual possibilities available to those making use of the example. In fact, one way of defining a knowledge community is by looking at its core inventory of examples, as such inventories form an essential part of the intellectual paradigm for any knowledge community.[3]

My work here is no exception to this truism that a scholar's choice of examples has fundamental effects for that scholar's intellectual work. Before embarking on a presentation of our first pair of propositions taken from Euclidean geometry, I will discuss briefly why I selected them. Such a discussion is to a certain extent a break with the standard form of mathematical presentation, which begins with a general observation about the importance of some new problem, invariably and substantially informed hindsight, and then follows with a sequence of definitions, examples, lemmas, a proof, and then some applications of the theorem. As Imre Lakatos argues in *Proofs*

[3]Kuhn, 1996, pp. 187–191.

and Refutations, this representation of the proof greatly obscures the actual process of proof-making, which is often most informed by the applications listed as an afterthought, and rarely settles the appropriate definitions until after the proof has been concluded to the author's satisfaction.[4]

My first motive for choosing these two propositions, and their associated proofs, is a biographical one. These were the first two formal mathematical proofs I memorized and was able to repeat. For me, they have continued to serve as archetypes for abstract mathematics, and were essential parts of my early familiarity with and intuition about mathematical proof. Beyond that, these two proofs consider properties of triangles which are widely taught to and taken for granted by young students wherever Western modes of mathematical pedagogy are in place. Moreover, the proofs make use of a suitably wide range of elementary geometric and algebraic techniques which require several degrees of abstraction and equivalence-making.

These two proofs are thus good representatives of a genre which might be called the *elementary mathematical proof*, consisting of short and conclusive expositions on more or less intuitively accessible concepts. Such proofs are popular for this sort of metamathematical analysis, whether by logicians or ethnographers,[5] because they are accessible to a more general readership and can be broken up into simple and concrete units of articulation. A more thorough piece would compare elementary proofs to those found in more advanced texts and papers. In the meantime, the reader will have to take me at my word that many similar phenomena continue in proof-making at higher levels of work, and indeed can become more pronounced as the concepts involved become increasingly technical or nuanced.

There is also a more general problem discussed in an appendix to Eric Livingston's "The Context of Proving" that expositions such as the one I am about to undertake take the form of demonstration by example. Common to sociological disciplines, and particularly to ethnomethodology, this style necessarily obscures the degree to which particular examples have been hand picked to make a particular point. Shapin notes that localist arguments using "idealized 'method stories' " are well suited to the goal of calling into question the enchantment and mysticism created by distance from the actual realm

[4]Lakatos, 1979, pp. 142–143.

[5]Livingston is perhaps the most widely cited of the few currently active ethnographers of mathematics. The introduction to Livingston, 1986, and Livingston, 1999 and 2006, frequently make use of elementary proofs as examples, and particularly elementary geometric proofs.

of scientific knowledge-making.[6] By selecting a small and intimate example of a particular mathematical practice, one can hope to shed light on its constructed character in a way which might be forbiddingly difficult if one were to instead address the discipline as a whole. At the same time, one risks focusing too narrowly on an insufficiently representative exemplar, and missing what is important, interesting, or essential to a field.

This preference for analyzing the particular and the mundane can also be understood within Schutz's *verstehende* program of sociology, which emphasizes contextual understandings of situated social phenomena.[7] While the mythos of a grand scheme of mathematics pervades reflections on the discipline, as well as much of mathematical writing, the fact remains that the context of teaching and proving is a highly local one, be it in a small tutorial or a grand lecture hall. There are always specific frameworks and tools used in specific expositions, producing a nuanced context-sensitivity with which any analysis must grapple. Importantly, this context-sensitivity does not necessarily preclude the possibility of generalizing example-driven arguments to broader spheres of practice. Rather, it must be remembered that across wide swaths of mathematical practice similar methods and approaches are used to incorporate and naturalize many varieties of context-dependent information. As I will discuss later on, how mathematicians present their work has a substantial affect on how it is received, both within and outside of mathematics, and similar methods of legitimation tend to sediment at the level of a profession, rather than at the level of a particular form of sub-disciplinary practice.[8]

It should also be noted that there is a substantial difference between discussing proofs already at hand and investigating the process of discovering them in the first place.[9] Although they do maintain many important similarities, there are marked differences between the *working conventions* involved in the production and the *performance conventions* permitting the presentation of proofs.[10] My focus here is on the presentational aspects of proof-making. I make no claims that all elementary proofs yield to the anal-

[6]Shapin, 1995, pp. 304–305.

[7]Heritage, 1984, pp. 45–46. Callon and Law, 1997, pp. 103–104 connect the *verstehende* program to agency and translation, both of which I discuss at length in earlier chapters.

[8]Abbott, 1988, pp. 184–185.

[9]Livingston, 2006, p. 41.

[10]In the context of art, see Becker, 1982, p. 63.

yses I will present in the same way these two do. Again, I must leave to the trust of the reader that, insofar as this analysis can be read methodologically, my approach can be extended to a wide range of discourses, both from within and without mathematics.

6.1.2 The Angles of a Triangle

I will begin by proving that the sum of angles in any triangle is equal to the angle of a straight line, which is variously quantified as 180 degrees or π radians. This method of proof is the constructive geometric method preferred for much of Western mathematical history, and valorized by Hobbes, Cauchy, and countless others. Before the widespread introduction of the formal algebraic rigor made possible by Cauchy, such geometric constructions had a virtual monopoly on legitimate proof practice.[11] The specific method of proof I will present was used in Proposition 32 of Euclid's *Elements*, which was a foundational model of early modern thinking about both mathematics and philosophy, particularly for Hobbes.[12]

> **Proposition.** The angles of a triangle sum to the angle of a straight line.
> *Proof.* Consider the scheme in figure 6.1. We denote angles by the set of

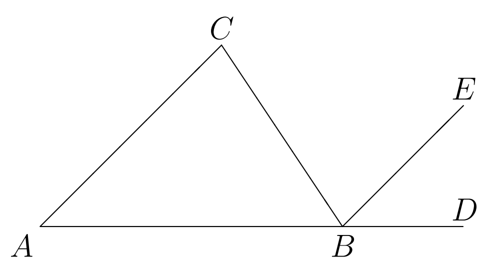

Figure 6.1: Construction for summing angles of a triangle.

vertices traveled to induce the angle. For instance, the leftmost angle of the triangle in the figure is denoted $\angle CAB$, as it is the angle formed by traveling from point C to point A to point B along the straight lines of the figure.

[11] Of course, there were varying extents to to which a given culture of proving demanded rigorous demonstration. See Livingston, 1999, and MacKenzie, 2001, for cultures of proving, and Richards, 2006, for a historical study of varying demands for mathematical rigor.

[12] Shapin and Schaffer, 1985, pp. 100–101.

Starting with an arbitrary triangle $\triangle ABC$, we begin by extending the line segment AB to some point D. Euclid gives earlier in the *Elements* a method of using a line and a point to construct a new line parallel to the original line and passing through this point. We use this method with the line given by the segment AC and the point B to construct the line segment BE, parallel to AC. We then make two applications of a theorem about angles formed when straight lines cross two parallel lines to obtain

$$\angle CAB = \angle EBD \qquad \text{and} \qquad \angle ACB = \angle CBE.$$

Thus, the three angles forming the straight angle $\angle ABD$ are the same as the three angles of the triangle. This completes the proof.

We should note here that figures accompanying this proof almost invariably have some explicit or implicit resemblance to the related figure 6.2, which shows a plausible arrangement of three copies of the 'same triangle' displayed in such a way that the three angles combine to form a straight line, as in the preceding proof. In the interest of full disclosure, I should note that I prepared this second figure first, and modified it to obtain figure 6.1. This second figure was also the one drawn for me when I first learned this proof. In this depiction, corresponding angles of the triangle are labeled with the same greek letter, according to a common convention of mathematical proof-writing which here reinforces both the relationship between a triangle and its internal angles and the claim that each of the three triangles apparent in the figure is indeed comparable to the others in the indicated way.

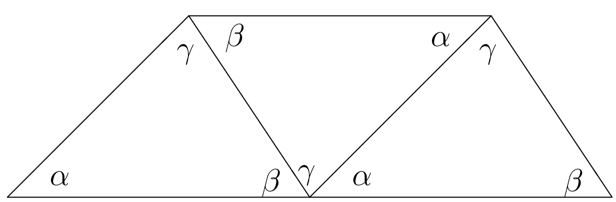

Figure 6.2: A related arrangement of triangles.

6.1.3 The Pythagorean Theorem

I shall now present a proof of the famous Pythagorean theorem that for a right triangle with legs of length a and b and hypotenuse of length c we have the relation

$$a^2 + b^2 = c^2.$$

This is not the proof given in Euclid's *Elements*, and in fact uses several techniques which would potentially be inadmissible in Euclid's constructive program as it was later formulated, and which would likely be incomprehensible to a Classical Euclidean mathematician. The Pythagorean Theorem is certainly among the most proved theorems in all of mathematics. Loomis, 1968, compiles 284 pages of proofs and demonstrations of the Pythagorean theorem divided into four general techniques.

Proposition. In a right triangle with legs of length a and b and hypotenuse of length c, we have

$$a^2 + b^2 = c^2.$$

Proof. We start with a right triangle with legs of length a and b and hypotenuse of length c. We can arrange four congruent copies of this triangle into a square as in figure 6.3. We know it is a square because the angles are all right angles and the sides are all of equal length $a + b$. We can then

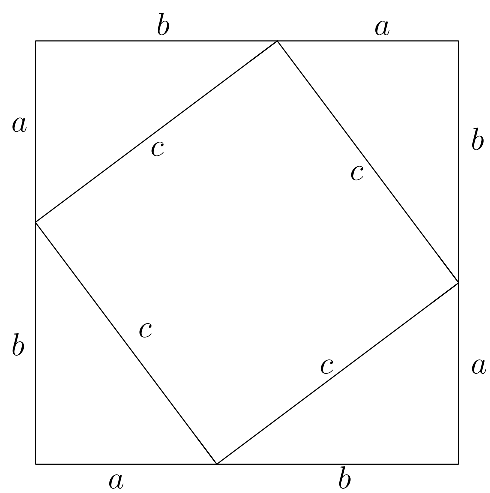

Figure 6.3: A square of four right triangles.

observe that the shape formed by the four hypotenuses is a square as well, since each side is of length c and the angles are all right angles, since they sum with the other two angles of our right triangle to a straight line. We can then determine the area of the larger square in two ways. First, we can square its side length to get $(a+b)^2 = a^2 + 2ab + b^2$. Second, we can add the area of the inner square to four times the area of each right triangle to get $c^2 + 4(ab/2)$. Thus,

$$(a+b)^2 = c^2 + 4\frac{ab}{2},$$

and we can multiply out and collect terms to obtain

$$a^2 + b^2 + 2ab = c^2 + 2ab,$$

from which we can subtract $2ab$ on both sides to get the desired identity

$$a^2 + b^2 = c^2.$$

This completes the proof.

6.2 Proof-Making

Let us state here a simple enough proposition with tremendous and far-reaching consequences: *proofs are made.* This can mean as little as the elementary idea that proofs such as the ones presented above had to be formulated, understood, and written. This can also mean that, since not every argument is a proof, there must be something to distinguish those arguments which are proofs from those which are not. To say that proofs are made is to say that whether or not any given argument is a proof depends on what properties and conditions are actively accorded to the proposition. A prover must claim her argument is a proof. An audience must buy in to the argument as a proof argument. A proof must be recognizable as such, and it is ultimately a body of social conditions which permits this recognition.

6.2.1 The Audience and the Distribution of Knowledge

The first major question of sociological importance concerning the process of proving is the matter of to whom the prover addresses her proof. There is another related question to which an answer might be more readily available. That is, who is expected to understand a given proof? As a reader of this work, for example, you are a participant in proof-making.[13] While I needn't expect all readers to understand every detail of the above proofs, this chapter at least very strongly carries an assumption that a typical reader would be able to follow the main structural features of the presented proofs. I expect, without needing to state as much, a certain basic linguistic and mathematical competence. This expectation is materialized in my presentation of the proofs, and it is also materialized in your reading of them, and in your recognition of them as proofs, and, moreover, as valid or meaningful proofs.[14]

There are very particular forms of geometric and presentational knowledge deployed in the above proofs. These techniques are expected to be familiar to most scholars in the university system who have partaken of a standard educational program, which includes, among other things, training in simple arithmetic and geometric computation and argumentation. There is a very clear biographical assumption pointing to the particular forms of general (or in the case of more advanced mathematics, where more formal training is assumed, specialized) knowledge.[15] This assumption is almost never stated explicitly, although it appears many times, particularly in textbooks of mathematics, in expectations of "a certain mathematical maturity" or familiarity with proofs.[16] The distribution of different levels of "mathematical maturity" is to a large extent common knowledge, and informs debates over what sorts of mathematics are appropriate for which students, many of which take place between people with only limited mathematical background themselves.[17]

[13]Whether you are an active or passive participant may depend on what you believe about authorship and literature, but I shall bracket this concern for my present purposes.

[14]Livingston, 1999, p. 870 points to the often overlooked fact that mathematical proofs are "designed for an audience—the practitioners of a mathematical culture—for whom that description is adequate."

[15]Schutz and Luckmann, 1973, p. 313.

[16]Lakatos, 1979, p. 142.

[17]Schutz and Luckmann, 1973, p. 312. It is beyond the scope of this work to examine the fascinating debates in post-war American mathematics pedagogy, where interested parties range in mathematical competence from relatively uneducated parents, to educated parents, to professional educators and education administrators, to professors of education, to professional and professorial mathematicians, and where mathematical and pedagogical competences are routinely invoked and challenged.

There is a slightly different form of biography at play behind specialized mathematical knowledge. As with all specializations, there is an educational structure implicit in the acquisition of such knowledge, but there is also a sense in mathematics that there should exist a certain mathematical reasoning faculty independent of a person's socialization. Counting and number systems are often regarded as cultural universals, independent of the society one observes.[18] This claim, which surfaces in many contexts from the history of mathematics to ethnology, obscures many essential differences between cultures, and certainly overstates the possibility of a mathematics independent of social organization or production.[19]

When claims to universality are made, it always pays to identify the particular people who make the claims. In the case of mathematics, such claims come from a wide variety of sources, including scholars, professionals, educators, and elements of the lay public. Mathematicians and teachers of mathematics have a vested interest in maintaining the idea of universal accessibility for mathematics, both to secure its status as the most natural of natural studies and to preemptively deny the possibility that their systems of logic and deduction come from a privileged and exclusionary position. Non-mathematicians also have a stake in mathematical universality, though it is certainly less obvious. Such a stake may come in the value of a regulatory ideal of achievable transcendence, or it may come in a form as mundane as the usefulness of the ability to trust users of mathematics, or an ability to rely on their own everyday arithmetical articulations.

This universality is rhetorically enforced, as we saw in chapter 4. Mathematical, and particularly constructive or quasi-constructive mathematical, proofs such as those presented above are usually written in the first person plural tense. Unlike typical scientific papers, proofs also tend to use more active than passive constructions. Where science obtains its truth-value from observing things as they 'just happen,' mathematical truth-value is obtained from deliberate manipulations of an ideal mathematical object by an active agent. Where the scientific first person is meant to bring the reader in as an observer (or in many cases simply to denote those conducting the experiment in question), mathematical first person discourse makes the reader a participant in constructing elements of the proof. Often, this explicit participation is codified by subjunctive grammar, which is virtually unheard-of

[18]Wilder, 1973, p. 32.

[19]Watson-Verran and Turnball, 1995, pp. 127–129.

in the natural scientific literature. In the first proof, "*We* begin by extending the line segment AB," a formulation which might also have been stated without the pronoun 'we' as a demand for the reader to become the agent of line extension. If part of the appeal of mathematical argument is its accessibility to everyone, then this form of reasoning is particularly effective, since it eliminates questions of technique and reproducibility found in science by making a proof something not only that everyone *can* do, but something that everyone *must* do.

Despite the explicit and implicit universalist pretensions of written mathematical proofs, the fact remains that the actual audience of mathematical proofs is incredibly limited. As with all disciplines, responsibility for the certification of mathematical knowledge falls to those with the specific established competences to certify it. Collins calls this group the core-set,[20] and MacKenzie points out their central role in evaluating new proofs,[21] especially those proofs with a high level of complexity or drawing from a range of established methods and conceptual areas. For most active problems in mathematics, there are very few working mathematicians with the requisite expertise to make a meaningful contribution to the body of work on that problem.[22] Moreover, for every mathematician capable of generating new results, there are generally only a few capable of critically evaluating them in full.[23] In this context, proof checking often amounts to checking local claims and heuristics, as well as examining the underlying structure of the argument, without necessarily attempting to examine the entire apparatus of the proof. Indeed, as was discussed in section 5.2.5 (page 118), there are many

[20]Collins, 1981.

[21]MacKenzie, 1999, p. 42.

[22]I have been told by a mathematics professor that one had to stay in one's area of specialization from graduate school until one earns tenure at a university, because it takes at least three years of study without producing any new papers to learn a new area.

[23]The system of peer review is premised in part on there being a sufficient supply of competent evaluators. Anecdotal evidence seems to support the claim that this premise is at best highly optimistic. While in the physical sciences it is assumed that a reviewer will not attempt to (or be able to, within the bounds of time and material resources) reproduce a paper's results before certifying it as valid, there is an assumption that the mathematics of a paper can and will be checked in a more direct manner. Adding to the conceptual difficulty and temporal demands of such a checking are the networks of gamesmanship, trust, and disinterest, all of which allow proofs to achieve the status of certified knowledge while never being subjected to the putative standards of disciplinary self-policing assumed to be in place.

mathematical works for which even the immediately presented material in support of a proof is incomprehensible on the full scale of the argument. And this does not even account for the large body of supporting work and justification which must be implicitly and explicitly cited in the materially present proof narrative, which is never checked and re-checked every time, or even any time, in which it is invoked. Proof checking, then, is sesquitextual.[24]

6.2.2 Examples and Geometric Abstraction

Proofs serve two principal social functions in mathematics. The first is to develop and test intuitions as a more formal way of 'comparing notes,' and the second is to provide a mnemotechnic system for integrating and organizing large swaths of mathematical information in a common and commonly accessible way.[25] In order to better fulfill these roles, proofs almost invariably make reference to familiar images as points of departure for purportedly rigorous analyses.[26] Moreover, the images are arranged in ways which emphasize the properties to be demonstrated. In the contemporary mathematical idiom, proofs from Euclidean geometry are rarely found without figures to illustrate the targets of notation, and these images both provide an important intuition for the proof and help one remember the proof technique. The representative images thus deserve particular attention.

In the second elementary geometric proof above, the triangle of record is regularly proportioned in such a way as to allow the reader to see which parts of each copy of the triangle correspond to each other, while still being able to see each individual triangle as a unified entity. This would not be the case, for instance, if the triangles were isosceles right triangles, where there would be no readily observable difference between the sides of length a and b, nor if b were much larger than a, in which case it would be hard to recognize the triangle as a familiar right triangle. In addition, the four triangles are arranged in such a way that the large square in the proof has sides corresponding with the vertical and horizontal axes of the page, which is the typical presentation of squares in mathematical diagrams. As a key step in the proof is recognizing that the triangles can be arranged to form two squares, the larger of which being harder to recognize, this presentation greatly aids the rapid assimilation of the proof's argument.

[24] See chapter 4, especially page 95.
[25] Lakatos, 1979, p. 29.
[26] Garfinkel, 1967, p. 35.

We might call the particular brand of exemplary triangle used above a *slightly scalene* example. A scalene triangle is one in which no two sides are of equal length. When one thinks of a triangle in the abstract sense, one of the first things likely to come to mind is an equilateral triangle, with all sides the same length and all angles of equal measure. Equilateral triangles are familiar and comforting, evoking associations both related to and distinct from their use in mathematics. Unfortunately, there is such a thing as too much order. While a prover needs the object in the figure to be rapidly diagnosable as a triangle, she also needs to maintain the conceit of generality in her observations. Most properties of triangles are trivially demonstrable for equilateral triangles (or, for right triangles, isosceles right triangles) because these triangles have a great deal of extra structure which an abstract, ideal, or general triangle doesn't have.

Thus, provers have to attain recognizability in their examples without transgressing into the realm of triviality. This dilemma is solved in virtually every mathematical text, including the one I have produced above, by using triangles where all three sides are observably of different lengths, but are all within a relatively close length of the other two. This is a clear example of the integration of common and discipline-specific conventions in mathematical symbolism in a manner which can also be found in many other disciplines.[27] The slightly scalene example purports to gain no special advantage as a privileged representative of the entire ideal class of Euclidean triangles.[28] In general, this implication about the proof-neutrality of an example is patently false.[29]

It is one thing to produce a slightly scalene triangle and have it stand in for all triangles, but what would a *slightly scalene continuous function* look like? As we shall see in section 6.3, Cauchy was certainly trying to make claims about the entire spectrum of continuous functions on the basis of the behavior of a few exemplary ones. Nobody could possibly consider each continuous function[30] at every point when proving a theorem about continuous

[27] Becker, 1982, p. 44.

[28] Related examples for inscribed angles are given in Livingston, 1999, p. 882, and Livingston, 1986.

[29] c.f. Livingston, 2006, pp. 39–40.

[30] One should not forget the crucial liaisons between the structure of exemplarity and the project of mathematical abstraction. There is a difference between saying a continuous function is one which 'looks like this' or 'is produced in this way' and defining continuous functions as Cauchy or his successors did. Definitions, too, have to vie for the status of

functions, but a well chosen such function can stand in for the rest. In mathematical pedagogy, a relatively small class of such examples, considering the many different contexts in which they are used, seems to dominate. Continuous or differentiable functions are smooth, closely bounded curves with a few local maxima and minima at x values not too far from zero. Discontinuous functions are usually continuous functions with one or more small jump discontinuities. Non-differentiable functions, are represented by drawings of continuous functions with varying types of cusps. From a still broader perspective, most counterexamples to theorems about specific mathematical objects occur in 'pathological,' or even 'teratological' cases where the shapes or objects at hand begin to look less and less like familiar or archetypal examples such as those which would be used in an illustration.[31]

Generally, geometric examples (or, moreover, though we will not be able to discuss them in such detail, all mathematical examples) are judged by the extent to which they portray the significant properties and potential obstacles in a proof or a problem.[32] To some extent, this gets the order wrong, at least in part, as paradigmatic examples play a central role in figuring what the important properties of a mathematical object might be. As for mathematical obstacles, since most of them occur in cases which are hard to represent, or, if representable, are hard to recognize as legitimate or relevant obstacles, examples always have a conflicted status when included in proofs. On the one hand, they often fail to portray one or more of the ways in which a proof technique can fail. On the other hand, without at least implicit reference to one or more concrete examples most geometric proofs become incomprehensible. It is therefore a crucial problem in mathematical proof-making to find a *valid* example, where validity is determined by "the degree to which [an example] measures the concept it purports to measure."[33] The words *valid* and *example* are in competition here, because the more valid a representation is, in the sense of conveying only what is understandable

legitimate stand-ins (see Lakatos, 1979), and those definitions are often judged by how they classify certain examples deemed either properly exemplary or teratological.

[31] There is an extensive discussion of such dismissive labeling throughout Lakatos, 1979. Hermite famously said "Analysis takes back with one hand what it gives with the other. I recoil in fear and loathing from that deplorable evil: continuous functions with no derivatives." Quoted in MacHale, 1993.

[32] For the proof-specificity of the properties of proof examples, see Livingston, 1999, p. 873.

[33] Eaton, 2001, p. 22.

through the abstract or ideal properties of an object, the less it appears as an example and the more it appears as a monster, or fails to appear at all.

Moreover, a valid example, because it retains an allegiance to its status as an example, must make itself seem dispensable to the proof. If a proof argument depended in some mathematically important way on the particular example chosen, it would not be a proof at all. A well chosen example must never admit to its fundamental importance in the process of proving. All this, despite the example's position as a *workable object* in the production of the proof. The messy reality of proving requires that examples be manipulable and capable of doing conceptual work by being the subject of manipulations *on behalf* of that for which each is an example. An example is truly a spokesperson in the Latourian sense.[34] This is so regardless of whether the example is geometric, diagrammatic, schematic, or algebraic. The trick is to make exemplary work invisible. The spokesperson must demonstratively speak with, and only with, the voice of those whom it represents. That this is impossible does not prevent it from conceptually undergirding every use of any particular example in mathematical proofs.

The central importance of making images through depicted examples can be understood within the larger dialectical system of society construction and reality maintenance. This system consists of cycles of externalization, objectivation, and internalization.[35] A specific notion of a triangle, such as one afforded by thinking of variations on an equilateral triangle, becomes typified and projected through the process of externalization as a model of an ideal triangle.[36] Through the rhetoric of the proof, and in particular the exemplary status of the example, the specific triangle-example becomes objectified as an ideal triangle. While mathematicians will acknowledge that they are only working with a particular example, operations such as those in the two proofs above are performed on one model triangle (which all the time maintains its identity as a single triangle, despite translations, rotations, and relablings) as though it were simultaneously all possible triangles. Only under this premise does the proof actually prove what it claims to prove. Lastly, because of its status as an objective, ideal object, the example triangle becomes internalized as such, and conceptually forbids other possible triangles as impostors. Indeed, counterexamples and counterintuitive

[34]Latour, 1999, p. 268. Latour, 1987, p. 71. See also page 33 of this work.

[35]Berger, 1967, pp. 13–14.

[36]c.f. Berger and Luckmann, 1966, pp. 30–34.

forms of an object often have to undergo decades of scrutiny before being regarded as legitimate counterexamples, and the status of an *ideal object* in mathematics is never a stable one.[37]

6.2.3 Algebraic Abstraction and the Logical Proof

Arguably, the difficulties described above do not apply, or apply in substantively different ways, to algebraic or logical proofs, and there have been important historical methodological splits between these two approaches.[38] The preceding chapters lay out a variety of grounds by which representational issues in elementary geometric proofs might apply more broadly to mathematical proofs in general. Here, I wish to sketch some of the socially manifested issues involved in coding geometric phenomena as algebraic ones.[39]

Both proofs offered above rely on meaningful and stable labeling and recombination of geometric angles. Labeling and summing angles are both practices with their own supporting conceptual foundations. Among those attendant notions common to these two practices are those of equivalence and comparability, and all of these foundational matters can be taken apart at the level of their practiced use in a proof procedure. The principal instance of algebraic coding in these proofs, however, is in the creation and manipulation of the notion of a *side length.* The concept of side length is taken for granted in my proof of the Pythagorean theorem. Two broad assumptions about the sides of the triangles in the proofs should be made explicit: those of their commensurability and generalizability.

The first fundamental assumption is that sides of a triangle are always comparable to any other line segment under consideration, whether one can visualize it or not, and whether or not one can construct an explicit comparison. This is by no means a trivial concern, and was indeed a central problem, and eventually a paradox, for Pythagorean mathematicians.[40] For members of the Pythagorean school of mathematics, the assumption that all line segments were commensurable meant that one could find a unit of measurement sufficiently small that it could, without residue, measure both line segments. As can be algebraically proven in less than a paragraph, this assumption

[37]Lakatos, 1979, especially pp. 136–139.

[38]c.f. MacKenzie and Wajcman, 1999, p. 18.

[39]Lakatos addresses such issues at length, albeit much more optimistically than I shall. Lakatos, 1979, p. 122.

[40]Wilder, 1973, pp. 88–90.

is false even for the side lengths of the unit isosceles right triangle, which measure 1, 1, and $\sqrt{2}$. By representing the length of a side by a letter such as a and then comparing it algebraically with other side lengths, named by the same or different letters, the mathematician assumes a commensurability which is independent of the location of the line segment and the actual form and representation of the lines themselves.

This move artificially generalizes the computation, invoking the second assumption hidden by algebraic representation. By labeling the lengths of sides a, b, and c instead of their actual lengths on the page, respectively 2.4, 3.2, and 4 centimeters, I have put forward the claim that whatever computations I do involving these lengths can be performed equally well with whatever side lengths one chooses, provided they correspond to sides of a right triangle. This computation would be independent of whether such a triangle can even be drawn or measured in the first place. Underpinning this operation is a sort of taxonomic conquest, whereby an object such as a side length is named, allowing any object with the same name to stand in its place. This strategy is widely used, in everything from recipes in the kitchen to stage directions in a theater, and represents in this case a leap to pure geometric ideality as an organizing idea.[41]

When I label the elements of figure 6.3, I practice a form of coding which pervades the professional vision of mathematicians. By coding, I declare what is important in the figure, how these important characteristics relate to each other, and how they can be understood within the broader context of geometric knowledge.[42] Algebra provides a way of giving an official representation of lengths in geometric objects in a highly theoretical manner.[43] This algebraic coding, however, operates at many theoretical levels, from elementary school arithmetic to formal symbolic logic, and each level creates from the coding its own scheme of legitimation.[44] Algebra becomes a central part of the universalizing narrative in the geometric proofs. We saw, in fact, that Cauchy's complaint with his predecessors such as Euler was that this universalization went too far. By calling for a new, disciplined universalism, Cauchy manages to obscure the universalizing pretensions of his own mathematical epistemology.

[41] Schutz and Luckmann, 1973, p. 320.
[42] Goodwin, 1994, p. 606.
[43] Berger, 1967, p. 30.
[44] Ibid., pp. 38–40.

6.2.4 Narrativization

Fundamentally, a proof is a story with a legitimating purpose for a mathematical idea.[45] As such, it is subject to the basic ethnomethodological requirement of account-ability, to use Garfinkel's coinage. The accounts found in proofs form a schema of objectivation and legitimation, and at the same time portray a story about how mathematics functions. Every mathematical idea has its storyable and mundane characteristics, and the choice of which aspects of an idea matter enough to be put in a proof shows definite, if not always deliberate choices about what matters in mathematics. The story-making of proofs not only reflects, but actively creates the world of ordinary or everyday mathematics as it recounts mathematical conquests past and present.[46]

It is no accident that mathematical proofs often involve some combination of rhetoric from stage directions and guided tours. The mathematical self is built from an explicit theatricality, in Goffman's sense of the term.[47] This theater of mathematics, through the process of abstraction, deals in typifications and institutionalizations of mathematical objects and ideas. These processes go a long way toward claiming objectivity for such constructs.[48] This, in turn, gives a basis of legitimacy to established mathematical practices.[49] One can trace a cycle of mutual establishment and reinforcement of discursive and theoretical motifs which go into the creation of the mathematical life-world.[50] Such world (re-)creation occurs at all levels and within all sub-disciplines of mathematics.

As a report, proofs operate as logical biographies for the theorems they describe. Their story often starts with conception from within a current shortcoming in mathematical understanding. The concept is then born through

[45]Livingston, 1999, p. 873 claims to be the first to directly comment on mathematical argument as a practice of narrative presentation. Alexander, 1995, argues that mathematical practice is a story-telling practice by considering its place in imperialist Elizabethan England. See especially p. 590.

[46]Sacks, 1984, p. 417. Livingston, 1999, p. 874 qualifies this view by arguing that the narrative that appears is not the proof itself, but a description of the proof. For a discussion of the effects of the story-making in mathematics on historiography in the case of Hamilton's Quaternions, see Koetsier, 1995, and Pickering and Stephanides, 1992.

[47]Holstein and Gubrium, 2000, p. 35.

[48]Eaton, 2001, p. 5.

[49]Ibid., p. 8.

[50]Berger, Berger, and Kellner, 1974, p. 63.

the statement of definitions, and takes its first steps in a series of simple examples and elementary lemmas. The concept comes into its own in one or two major theorems, and sails off into the sunset of applications and open problems involving the concept, or to which the concept might be applied. Like all biographies, this one has an agenda. It represents an idealized form of mathematical production of which Lakatos is highly critical. In this narrative, math is an orderly, deterministic process with clear foundations and objectives, much as it is understood in popular mythology and understandings about the distribution and operation of mathematical knowledge. Such proof biographies have a double structure, for they are simultaneously narratively and logically ordered.[51]

Perhaps surprisingly, this biography is the result of a series of reconstructions which would rival any account of conversion.[52] While there is no single process of mathematical discovery, it is almost universally true that the many aspects of the above presentation do not come in the prescribed order, and indeed are co-created side by side as mathematical intuitions and notions are developed and translated into the language of proofs. Most often, proofs begin in the interplay between applications and examples, which give both an impetus and an intuition for the proof. Naïve definitions are then proposed, and a process of continual negotiation between definitions, examples, lemmas, and theorems results in a plausible system of statements which can be published as a mathematics paper or presented in a lecture or colloquium, or even a homework assignment, as this process happens at many different scales of time, investment, and human participation.[53] Between the first two sentences of the second proof above is a vast amount of mathematical reasoning and experimentation which has been reduced to a single dot and a space. Before the problem has even been posed, it is saddled with a long and convoluted history.

Even this modified account imposes an ideal of mathematical production on a narrative. Such rigorous dialecticism is hard to miss in a close reading of Lakatos, and he goes so far as to make it explicit in his own proposal for heuristic proof-accounting.[54] One invariably encounters the problem of having to privilege one or more specific and local discourses from which to

[51]See section 5.2.3.

[52]Snow and Machalek, 1983, p. 266.

[53]Lakatos, 1979, and Rosental, 2003, offer, respectively, historical and contemporary case studies in this process of negotiation and certification.

[54]Lakatos, 1979, pp. 144–154.

compare the narratives of others to the ever elusive 'Archimedean point' of right knowledge and right method.[55] One methodologically defensible position with respect to the problem of the Archimedean point is to acknowledge the historicity and situated character of all proof-concepts and their corresponding proof-narratives, and to bracket the accounts accordingly.[56]

6.2.5 The Visual Practice of Mathematics

The narrative presentation of mathematics structures much of the proof process and the ability of mathematicians to communicate and establish mathematical ideas, and it is tempting to leave our analysis at that. But it would be a grave error to leave the discussion at that without attending to the material form these narratives take, and the material contexts within which these narratives are elaborated. A close analysis of the work of mathematical communication reveals it to be, by and large, a *visual practice*. This visual structure is evident in everything from the material written (and often diagrammatic) presentation of mathematical proofs, to the chalkboard work of proving and teaching, to the visual language of mathematical exegesis. That mathematicians can claim that mathematics itself is essentially non-visual is a testament to the success of this visual practice in establishing itself as epiphenomenal.[57]

Of course, one could immediately cite the success of visually blind mathematicians to counter this claim. It is true that mathematics is not uniform in its modes of practice, and work from outside of its dominant modes may certainly gain acceptance and be certified within the larger disciplinary establishment of mathematics. Powerful regimes of translation, which allow alternative modes of mathematical production to be translated into the algebraic and logical idioms of official mathematical proof, permit one to gloss over the fundamentally visual way by which official mathematics is proved, certified, and communicated. Since I am concerned here with mathematics as a disciplinary community, I have chosen to focus on dominant community-

[55] Shapin, 1995, pp. 313–315.

[56] David Bloor's *Knowledge and Social Imagery*, 1976, which elaborates his 'Strong Program' in the sociology of knowledge, represents a much discussed and worthy attempt at such a framework. The heterolingual formulation of mathematics at the end of chapter 5 is another possible framework which I propose deserves a similar place in the annals of the philosophy of mathematics.

[57] Rotman, 2003, discusses epiphenomenality in mathematical diagrams and simulations.

wide modes of mathematical production, which happen to be predominantly visually mediated. At the same time, attention to those means of production at the center of mathematics sheds important light on the conditions of legitimacy for peripheral practices, and so I am certainly not ignoring the arguably non-visual mathematics of, among others, blind mathematicians.

The most obvious locus of visual production in mathematics is in the written proof, examples of which can be found throughout this text. Livingston argues that a mathematical proof, as such, emerges like a *gestalt* image from the text and diagrams from the page.[58] Thus the mathematical proof is constituted in two ways: first, as a physical and visual record, and second as an emergent argument rooted in this material record.[59] The emergent argument reads the whole body of experience impinging on the mathematician into the proof she is reading, allowing it to claim to escape its concrete written materialization. Yet the materiality of the proof always returns. It returns in the space limitations which constrain argumentative possibilities, in the requirement for notational clarity (and at the times when an unfortunate notation confuses or obscures an argument), in the workable diagrams which help to constitute an argument, and in the moments of surveying a text when local arguments are comprehended. It returns also where the visual presentation of an argument supersedes its formal claim-making. In fact, many geometric demonstrations employ what Livingston calls *observable logic*, to which he opposes formal logic, where a mathematical statement is observably true, and so beyond dispute.[60]

By materializing the conceptual elements of the proof, a proof argument makes the proof tangibly witnessable.[61] One can speak of theory itself becoming materialized or being materially-generated, particularly through the use of diagrams.[62] This phenomenon is not limited to journal articles and

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[58]Livingston, 1999, especially pp. 868–869. See also Wittgenstein, 1963, II: xi.

[59]These two modes interact in many ways, as I have attempted to show throughout this work. One example comes where a proof involves reading a mathematical statement in two ways, as in the proof that the identity element of a group is unique: write the two identity elements as x and y; multiplying them together gives $x \cdot y$, which can be read either as an identity multiplied by y or an identity multiplied by x, giving $y = x$. Livingston, 1999, p. 872. Both elementary geometric proofs given at the start of this chapter similarly involve seeing, respectively, an angle or a square in two ways.

[60]Ibid., p. 872.

[61]Livingston, 1999, p. 874. Such proofs then exist in what Rotman calls the *picturable world*. Rotman, 2003.

[62]See Galison, 1997, and Kaiser, 2005.

textbooks. Indeed, blackboard presentations of proofs and blackboard writing offer perhaps the best example of mass witnessing in mathematics.[63] At the blackboard, the process of mathematical inscription is literally performed, and combined with speech and gesture to elaborate and emphasize mathematical statements as they are combined and recombined to produce a proof argument.[64] As in Boyle's experimental demonstration space at the Royal Society of London, a mathematical classroom or lecture hall can be transformed by the inscribed and gestural work of mathematical proving into a public (or, rather, quasi-public, since the same considerations of physical and conceptual access apply in these spaces as applied for Boyle) space where groups of qualified witnesses can be amassed to give assent to mathematical propositions. This is equally true for colloquia and seminars organized to discuss new results as it is for mundane classroom instruction organized to present established truths.

On paper and in the classroom, mathematical results are presented as witnessable entities whose truth value emerges from the amassing of either solitary witnesses at their desks or gathered bodies in front of a blackboard. In either case, the visual communicative format of mathematics serves to collect and deploy qualified assenters to the truth and validity of mathematical propositions. It is the visual culture of mathematics that allows synoptic and gestural apprehension of mathematical phenomena. It allows the physical sedimentation (in the form of writing) and desedimentation (in the form of comprehending) of mathematics which is essential for producing assent in a way which can be shared and universalized. That is to say, it is the visual culture of mathematics which makes mathematics possible.

6.3 Cauchy's Intermediate Value Theorem

I would like to close this chapter with a consideration of two proofs of the Intermediate Value Theorem from Cauchy's 1821 textbook, *Analyse Algébrique*. My reading of these proofs in this section[65] begins with the idea that rigor is socially achieved,[66] and then asks how this social achievement is encoded in

[63]See Livingston, 1999, p. 873.

[64]See Greiffenhagen and Sharrock, 2005.

[65]See section 3.2, page 62, for a discussion of these proofs in terms of Cauchy's geometric intuition.

[66]MacKenzie, 2001, p. 12.

Cauchy's written proof. One would expect to find evidence of much of the social work of rigor production in Cauchy's written text, both because it was written to function within the social space of rigor which Cauchy was enunciating, and because the spread and mass socialization of Cauchy's program of rigor was in large part by means of the diffusion of his written work in mathematics.

6.3.1 A Proof by Cartesian Intuition

The Intermediate Value Theorem is stated and proved quite early in Cauchy's presentation of continuous functions, which begins in the second chapter of the text after his first chapter introduction and framing of the function concept. As a theorem about continuous functions, the Intermediate Value Theorem says precisely what a reasonable intuition about such functions would have one believe: that a continuous function doesn't skip anything as it goes from point A to point B. Stating and proving the theorem at all was a new practice in the world of mathematics at the time of Cauchy's writing. Continuous functions were ordinarily assumed, at times in a way verging on axiomatic, to have this obvious property. One reason why they could be assumed as such was because the set of exemplary functions for the concept of continuity were largely drawn from problems of modeling concrete mechanical situations where the continuity of space and motion could be taken entirely for granted.

Others had certainly stated and offered explanations of the Intermediate Value Theorem before Cauchy, but Cauchy was among the first to base his justification of the theorem on an explicitly stated analytic notion of continuity, one which has lasted until the present day in mathematical analysis. It is quite likely, interestingly enough, that the attempt to analytically formulate what it meant for a function to be continuous was the principal enabling factor for a vast expansion in the range of exemplary continuous functions over the next century and a half. This ever-increasing stock of examples would fundamentally change the intuitions and procedures surrounding continuous functions in professional mathematics. Examples which are commonplace now, such as the Cantor-Lebesgue function developed at the turn of the twentieth century, would be teratological or unimaginable to Cauchy's contemporaries and predecessors. At the same time, commonplace continuous functions in the decades before Cauchy's text, including several varieties of step functions, would have hard time passing muster under present day mathematical criteria for continuity.

By placing his discussion of the Intermediate Value Theorem early in his text and shortly after his definition of continuity for functions, Cauchy makes a corresponding positional statement for the theorem within the broader corpus of mathematics. Cauchy's narrative ordering establishes a logical ordering which puts analytic definitions first and hitherto obvious theorems second. In Cauchy's new framework, the obvious is derived from the precise, and the precise needn't be obvious in itself, so long as it is capable of exact justification.[67]

Cauchy's logical ordering says one thing and appears to do another. Having placed the Intermediate Value Theorem after his definition of continuity for functions, one expects Cauchy to invoke this definition, or later derivations from this definition, in presenting his proof. For his first proof, the proof in the body of the text, this is precisely and conspicuously what Cauchy does *not* do. Instead, Cauchy's first proof implicitly draws on the standard exemplary visualization of continuous functions which had pre-existed his work of rigorously defined continuity. Using the model of a graphed continuous function whose vertical coordinates represent the values assumed by the function, Cauchy's proof reduces to arguing that if a function is continuous and goes from one value to another, it must cross any given horizontal line between those two values, and thus must assume that value.

The author, himself, admits that this proof is analytically inadequate to the entire range of purposes at which he aims, and he offers at the end of his first proof to give a "purely analytic" proof in an appendix. We should, however, note something Cauchy does not say, which would later be put in his mouth by adopters and apologists for his program of rigor. Cauchy certainly never says that the proof provided in the body of his textbook fails to meet the standard of rigor he is attempting to establish, though it certainly would so fail if presented to a present day mathematician trained in Cauchy's tradition. What is important for Cauchy is that the proof maintains an allegiance to elementary concepts of quantity and relation from plane geometry, which by Cauchy's time included the Cartesian coordinatization of the plane which serves as the basis of graphing functions.[68] Indeed, the proof

[67]Russell and Whitehead, 1950, pp. v–vi, take an extreme form of this methodological precept, aiming to "prove as much as is true in whatever would ordinarily be taken for granted," building from premises whose aim is not to be obvious or ordinary but rather to be consistent and fundamental.

[68]For Descartes's merging of algebra and geometry, see Grosholz, 1980.

relegated to the appendix, which would be front and center in virtually any contemporary incarnation of Cauchy's course, may have been so relegated precisely because it introduces what for Cauchy would be an unnecessary layer of abstraction and removal from the problem's geometric origins. What Cauchy notes about the second proof is that it is advantageous in furnishing a numerical solution, not that it is in any way more precise or more rigorous. As we saw in chapter 3, numerical solutions and approximations were definitely not pedagogical priorities of Cauchy's.

6.3.2 A Proof by Numerical Approximation

Buried as a later corollary to the first theorem in the third appendix to Cauchy's text, Cauchy's second demonstration of the Intermediate Value Theorem is the one which seems to have best survived the appropriations and rearticulations of Cauchy program of rigor in the nearly two centuries since it was proposed. The preliminary theorem Cauchy uses to establish the Intermediate Value Theorem states that if the values of a function are of different signs at the endpoints of an interval, then the function must have changed signs at some point in the interval, and in so doing must have attained the value zero at some point.

The proof consists of dividing the interval in question into m parts, in a technique which would not be unfamiliar to engineers studying surveying at the École where Cauchy taught. Without further explicit justification, Cauchy claims that upon dividing up the original interval one must necessarily find a change of sign in at least one of the new subintervals. Taking one such subinterval and dividing it into m parts, one finds an even smaller interval in which a sign change occurs. This process repeats as one effectively analytically zooms in on the site of the sought-after sign change.

Cauchy invokes his definition of continuity by citing the first theorem derived from it in the main text, which states that if one takes a sequence of points converging to a limit and examines the values of a continuous function at these points, the function, too, will converge to the value of the function at the limit point. As the endpoints of the intervals of sign change converge to some point a, and as the value of the function must have opposite signs at either endpoint of the interval, Cauchy claims to establish that the value of the function at a is simultaneously both positive and negative. Since such examples as the function $f(x) = x^2 \sin(1/x)$ would not have been part of the standard exemplary inventory for Cauchy, one can understand his lack of attention to describing the sign changes of the function where a present day analyst would spend some effort on regulating the signs of individual terms in the sequence. We should also observe that Cauchy, while he frequently used the language and terms now associated with logical quantifiers (such as *there exists*, *for all*, or *if. . . then*), his work, at least in the period we are considering, predated the development of formal methods and rules for manipulating these quantifiers in a consistent and proof-certifying way which protects the validity of analytic assertions.

It is important to note that, while some things are explicit in the second proof which are not explicit in the first, there still remains much which is left to the logical, and even geometric intuition of the reader. For instance, that if a sign changes over an interval it must necessarily change in some sub-interval is not a difficult claim to justify, but neither is it true that such a claim is so trivial as to be beyond the possibility of a few sentences of explanation, even within the mathematical vocabulary of Cauchy's day. Cauchy's use of the limit concept is at times deleteriously intuitive, and this is especially clear in his treatment of converging sequences where corresponding elements have opposite signs. Again, his level of detail would not have been problematic within the paradigm of examples from which he drew, although more would almost certainly be required in a present day course in analysis. The purpose of the second demonstration seems not to be, as one might suppose, to give a presentation which is in some way more rigorous than the first. Rather, it is in many ways a more vulgar analytic approach to the problem which merits mention almost entirely on the grounds of its potential practical utility, and it is given in a section which may well have been added, like his discussion of infinitesimals, to the end of the text in order to please his pedagogical critics.

Thus, the structure, language, and content of Cauchy's mathematics depended integrally on the social community of mathematicians and administrators within which he was working. The effects of these relationships shows itself in both the placement and the presentation of Cauchy's two proofs of the Intermediate Value Theorem. These proofs enact a politics of logical ordering, as well as a stance on the proper modes of mathematical reasoning and the proper ends of mathematical production. Furthermore, while the social circumstance of Cauchy's work affected its presentation, it must also be observed that Cauchy's presentation went a long way toward enacting and shaping the disciplinary community in which he worked. Looking out from the text, one can see the sedimentation of a space of practice through its means and modes of articulation.

6.4 This Completes the Chapter

Mathematics is, like any discipline, a product of its presentation. A proof is no mere demonstration of a mathematical surety. Rather, it is a complex system of representation which engages many levels of narrativity and which tells a very specific and situated story about mathematical production. Instead of being a closed system of ideal signifiers whose play is discernible and controllable, mathematics involves the constant negotiation of competing ideas and methods, none of which are ever entirely formed or understood, and certainly none of which could ever exist in a conceptual vacuum. If proof is to be understood as an authoritative assertion about the ideal nature of some pure system of concepts, then the biggest lie comes at the end of every formal mathematical argument. Sometimes it is written as a box, darkened or empty, or denoted by the letters QED, for the Latin *quod erat demonstrandum*,[69] or perhaps in the deceptively simple sentence

> This completes the proof.

[69] 'That which was to be shown.'

Chapter 7

Conclusion

Throughout this text, I have claimed that mathematicians are witnesses and proofs are testimony. I am now in a position to collect and synthesize just what this means. To make the claim that mathematics as a practice depends fundamentally on witnessing, I have found it necessary to broaden both what is meant by mathematics and what is meant by witnessing. Contrary to most widely circulated views of mathematics, I have taken mathematics to include the entire network of mathematical production which makes mathematical theorems and mathematical practices possible. And contrary to most widely circulated views of witnessing, I have taken witnessing to include the entire range of productive and testimonial practices which make witnessing possible.

If I have had to cast my net widely in order to pull together a discipline and a practice which I see to be integrally connected, it is because the discipline of mathematics and the practice of witnessing are both tremendously successful at isolating themselves, and making themselves appear simultaneously more and less than they really are. Mathematics makes itself appear more than it is by invoking a virtual realm of mathetic activity to which it claims perfect access but at which it can never do more than evocatively gesture. It makes itself appear less than it is by excluding as epiphenomenal any aspects of mathematical production which do not fall strictly under the rubric of explicit logical exegesis. Witnessing makes itself appear more than it is by claiming unmediated access to facticity and unambiguous representational power through testimony. It makes itself appear less than it is by naturalizing the status of the witness and shutting out the socio-epistemological relations which enable testimony to demand assent.

Witnessing enters into mathematics in a variety of ways and at a variety of levels. First, it is present in the testimonial structure of proofs. By pulling together relations of social authority and credibility with circumstantial factors surrounding the development and presentation of results, mathematical proof narratives quietly and obliquely invoke a powerful politics of social positioning in order to lend credence to results. A mathematical proof evokes a having-been, a having-seen, and a having-witnessed in the prover which is then shared virtually through the proof-testimony with the prover's audience. Following established presentational conventions, the proof makes the subject matter present to the audience, and performs the proof-ness of the proof by constant incorporation of and allusion to rigorous practices.

As a proof is being produced and articulated, mathematicians act as conscious and deliberate witnesses by performing their objectivity and right method. Augustin-Louis Cauchy's prescriptions and proscriptions in his Cours d'Analyse made a particular point of constantly policing the ways in which mathematical truth was accessed and expanded. Later mathematicians followed Cauchy's lead in defining precise formal mathematical practices which could guarantee the prover's allegiance to the right brand of mathematical ideality.

This performance of mathematics has more to do with what is invoked and not said than with what is explicitly worked in detail. Sesquitextual forms in mathematical proof suggest a more general sesquitextuality in the grand phenomenon of witnessing, and perhaps in language as a whole. That is not to say that what is said is immaterial. I have discussed at length how the choice of exemplars and motifs for presenting mathematics fundamentally shapes what mathematics is possible, meaningful, and productive. A successful proof effaces the locality of its work by means of a manufactured exemplarity which does away with the always mediated allusive form of any concrete argument.

For mathematics, the consequences of this analysis may be limited. On this front, my work has been far more descriptive than normative. I can hope that one effect of this study is to denaturalize the origins, form, and function of mathematical rigor in the everyday practice of mathematics. By thinking of mathematics first as a social formation, and only second as a disciplined mode of knowledge production, mathematics may well be able to better prioritize its methods and conventions to dispense with what may be a superfluously sanctimonious rhetoric of truthfulness. But I write this with hesitation. After all, it is likely that it is precisely these structures, systems, and conventions that have allowed a remarkable flourishing in mathematics to take place, and have permitted math to exist as such a heterogeneous enterprise, despite its homolingual foundations.

My work here has been more normative and may well have more meaning and do more good for those who would attempt to understand the position of mathematics within the culture of its production, as well as those who would rather exempt mathematics from such local contextualizations. I aim my conclusions at those ethnographers, sociologists, historians, and theorists who would make a critical study of mathematics, and at those who wouldn't dare to do so. In these disciplines, as in any, there are always problems in evaluating the effects of privileged discourses, for their very privilege makes them difficult to bracket. The ascendence of a mode and theory of knowledge which depends on producing logical certitudes can be seen as one instance of a broader tradition of logocentrism in Western metaphysics, where Western metaphysics itself is a coherent stance of articulation only through the regime of homolingual address. This regime is largely responsible for the apparent critical impenetrability of mathematics, and for its exceptional status among the scholarly disciplines.

There is a persistent counter to the critique of logocentrism, however. It can be summed up by the question: what if there actually *is* an underlying logic to things? What if centuries of thinkers have been operating under ultimately correct, if sometimes misplaced, assumptions of a basic unity in thought and discourse? These claims are often readily answered in careful analyses of literary and political discourse, and the last half century has seen a remarkable depth of criticism in these areas. Harder to circumscribe are arguments turning to the natural sciences, nature itself, and mathematics. Strikingly often, theorists make exceptions in their arguments for mathematical and logical thought, and strikingly often the false edifice of mathematical ideality is allowed to stand unassailed and even unregarded. Any serious critical encounter with the normative regimes of logocentrism and homolingual address cannot ignore what transpires in the last and most vigorously defended stronghold of epistemic ideality.

If there has been any one thesis running through this entire work, it is this: mathematics works, but it does not work on its own. Mathematics is the result of the *lived work* of real social communities of practitioners. Like any discipline, mathematics is supported by a system of practices which derive less from any originary epistemic transcendence than from the constant negotiation of practices of social and epistemic organization and regulation. A proof really does depend on how you put it, and the proof of the putting is in the practice of mathematics that results.

Appendix A

Translations from *Analyse Algébrique*[1]

A.1 Introduction[2]

[page j] Some people, who have wanted to inform my first steps in the scientific profession, and among whom I would cite with gratitude Messrs *Laplace* and *Poisson,* having testified to the desire to see me publish the Course in Analysis from l'École royale polytechnique, I have decided to set to paper this Course for the greater utility to students. I here offer them the first part known by the name *Algebraic Analysis,* and in which I successively treat several types of [page ij] real or imaginary functions, convergent or divergent series, solutions of equations, and decompositions of rational fractions. Speaking of the continuity of functions, I have been unable to dispense with making the principal properties of infinitely small quantities known, properties which form the basis for infinitesimal calculus. Ultimately, in the preliminaries and in some notes placed at the end of the volume, I have presented developments which could be useful to Professors and to Students of

[1] Augustin-Louis Cauchy. *Cours d'Analyse de l'École Royale Polytechnique.* 1re *Partie. Analyse Algébrique.* Éditions Jacques Gabay, 1989. The 1989 edition is a faithful reproduction of the 1821 first edition of the textbook, which was to be the first of a multi-part series which was never continued. The translation is my own. For the *Introduction*, I have drawn from partial translations by Iacobacci (1965), Richards (2006), and Ragland (Belhoste, 1991), and Stephanie Kelly (Cornell University) provided valuable editorial assistance.

[2] Cauchy, 1821, pages i–viii.

the Royal Colleges, and to those who would make a special study of analysis.

Regarding methods, I have sought to give them all the rigor one requires in geometry, in such a way as to never resort to reasons drawn from the generality of algebra. Reasons of this type, although commonly admitted, above all [page iij] in the passage from convergent to divergent series, and from real quantities to imaginary expressions, cannot be considered, it seems to me, but as inductions so as to sometimes apprehend[3] the truth, but which agree little with the vaunted exactitude of the mathematical sciences. One must likewise observe that they tend to attribute to algebraic formulae an indefinite extension, whereas, in reality, most of these formulae subsist only under certain conditions, and for certain values of quantities they contain. By determining these conditions and these values, and in fixing in a precise way the meaning of the notations of which I avail myself, I make all uncertainty disappear; and so the different formulae no longer present anything more than relations between real quantities, relations which are always easy to verify [page iv] by the substitution of numbers for the quantities themselves. It is true that, to stay constantly faithful to these principles, I have seen myself forced to admit several propositions which perhaps appear a little harsh at first. For example, I articulate in chapter VI, that *a divergent series has no sum;* in chapter VII, that *an imaginary equation is only a symbolic representation of of two equations between real quantities;* in chapter IX, that, *if the constants or variables comprised in a function, after having been supposed real, become imaginary, the notation with whose aid the function finds itself expressed, cannot be conserved in the calculus but by virtue of a new convention so as to fix the meaning of this notation in the last hypothesis;* &c. But those who will read my work will recall, [page v] I hope, that the propositions of this nature, entailing the happy necessity of putting more precision in theories, and supplying useful restrictions on over-extended assertions, turn to the benefit of analysis, and furnish several subjects of research which are not without importance. Thus, before carrying out the summation of any series, I have had to examine in which cases the series can be summed, or, in other words, which are the conditions of their convergence; and I have, to this end, established general rules which appear to merit some attention.

For the rest, if I have sought, on the one hand, to perfect mathematical analysis, on the other hand, I am far from pretending that this analysis

[3] *faire pressentir,* make sensible.

should have to suffice for all the rational sciences.[4] Without a doubt, in the sciences which one calls natural, the only [page vj] method which one should be able to employ with success consists of observing events and then submitting the observations to calculation. But it would be a grave error to think that one only finds certitude in geometric demonstrations, or in the testimony of the senses; and although no person until now has tried to prove by analysis the existence of Auguste or of Louis XIV, every sensible man will agree that this existence is as certain for him as the square of the hypotenuse or *Maclaurin's* theorem. I will say more; the demonstration[5] of this last theorem is comprehensible to a small number of minds, and the savants themselves aren't all in agreement on the extension one should attribute to it; whereas everyone knows very well by whom France was governed during the seventeenth century, and that no reasonable disagreement can be raised on the subject. That which I say here [page vij] of historical fact can equally apply to a multitude of questions, religious, moral, and political. We must thus be persuaded that there exist other truths than the truths of algebra, other realities than sensible objects. We cultivate with ardor the mathematical sciences, without wanting to extend them beyond their domain; and we won't imagine that one can attack history with formulae, nor give for moral sanction[6] theorems in algebra or integral calculus.

In closing this Introduction, I cannot dispense with recalling that the insights and advice of several persons have been very useful to me, particularly those of Messrs *Poisson, Ampère* and *Coriolis.* I owe to the latter, among other things, the rule on the convergence of products composed of an infinite number of factors, and I have benefited several times from the [page viij] observations of Messr *Ampère,* as well as from methods he develops in his Lessons in Analysis.

[4] *sciences de raisonnement*, sciences of reasoning.
[5] *démonstration* continues to translate 'proof' in present day mathematics.
[6] *donner pour sanction à la morale*; some translate this as 'base morality on.'

A.2 Table of Contents[7]

[7] Cauchy, 1821, pages ix–xiv.

Notes on Algebraic Analysis

End of table.

A.3 Two Proofs of the Intermediate Value Theorem

A.3.1 Statement of the Theorem[8]

$4.^{th}$ THEOREM. If the function $f(x)$ is continuous with respect to the variable x between the limits $x = x_0$, $x = X$, and if one designates by b an intermediate quantity between $f(x_0)$ and $f(X)$, one will always be able to satisfy the equation

$$f(x) = b$$

by one or several real values of x included between x_0 and X.

A.3.2 First Proof[9]

DEMONSTRATION. To establish the preceding proposition, it suffices to show[10] that the curve which has for its equation

$$y = f(x)$$

meets one or several times the level[11] which has for its equation

$$y = b$$

in the interval comprised between the ordinates which correspond to the abscissas[12] x_0 and X: but evidently it's that which will take place in the agreed-upon hypothesis. In effect, the function $f(x)$ being continuous between the limits $x = x_0$, $x = X$, the curve which has for its equation $y = f(x)$, and which passes 1^{st} through the point corresponding to the coordinates $x_0, f(x_0)$, 2^{nd} through the point corresponding to the coordinates X and $f(X)$, will be continuous between these two points: and, as the given constant b of the level which has for its equation $y = b$ finds itself included between the ordinates $f(x_0), f(X)$ of the two points which one considers, the level will necessarily

[8] Cauchy, 1821, p. 43.

[9] Cauchy, 1821, p. 44.

[10] *faire voir*, to make to see.

[11] *droite*, straight, right, or level line.

[12] Ordinates and abscissas are, respectively, y and x coordinates of a point. Cauchy uses the terms *ordonnées* and *abscisses*.

pass between these two points, that which the above mentioned curve cannot but meet in the interval.

One can, for the rest, as one will do it in note III, demonstrate the 4^{th} theorem by a direct and purely analytic method, which itself has the advantage of furnishing the numerical solution of the equation

$$f(x) = b.$$

A.3.3 Second Proof[13]

1.st THEOREM. *Let $f(x)$ be a real function of the variable x, which remains continuous with respect to this variable between the limits $x = x_0$, $x = X$. If the two quantities $f(x_0)$, $f(X)$ are of opposite signs, one will be able to satisfy the equation*

$$(1) \qquad f(x) = 0$$

for one or several real values of x comprised between x_0 and X.

DEMONSTRATION. Let x_0 be the smaller of the two quantities x_0, X. We make

$$X - x_0 = h;$$

and we designate by m any whole number greater than unity.[14] As with the two quantities $f(x_0), f(X)$, one is positive, the other negative, if one forms the sequence

$$f(x_0), f(x_0 + \frac{h}{m}), f(x_0 + 2\frac{h}{m}), \ldots f(X - \frac{h}{m}), f(X),$$

and successively compares the first term with the second, the second with the third, the third with the fourth, &c..., one will necessarily end by finding one or several times two consecutive terms which will be of opposite signs. Let

$$f(x_1), f(X'),$$

be two terms of this type, x_1 being the smaller of the two values corresponding to x. One will evidently have

$$x_0 < x_1 < X' < X$$

[13]Cauchy, 1821, pp. 460–463.

[14]Unity is a special term for the number 1 with many context-specific connotations in mathematics.

and

$$X' - x_1 = \frac{h}{m} = \frac{1}{m}(X - x_0).$$

Having determined x_1 and X' as one comes to say it, on will be able to do the same, between the two new values of x, placing two others, x_2, X'' which, substituted into $f(x)$, give results of opposite signs, and which should be proper[15] to verify the conditions

$$x_1 < x_2 < X'' < X',$$

$$X'' - x_2 = \frac{1}{m}(X' - x_1) = \frac{1}{m^2}(X - x_0).$$

Continuing thusly, one obtains, 1.st a series of increasing values of x, to know,

$$(2) \qquad x_0, x_1, x_2, \&\text{c.} \ldots,$$

2.nd a series of decreasing values

$$(3) \qquad X, X', X'', \&\text{c.} \ldots,$$

which, surpassing the first quantities respectively equal to products

$$1 \times (X - x_0), \frac{1}{m} \times (X - x_0), \frac{1}{m^2} \times (X - x_0), \&\text{c.},$$

will finish by differing in these first values as little as one will like. One must conclude that the general terms of the series (2) and (3) converge toward a common limit. Let a be this limit. Since the function $f(x)$ is continuous between $x = x_0$ until $x = X$, the general terms of the following series

$$f(x_0), f(x_1), f(x_2), \&\text{c.} \ldots,$$

$$f(X), f(X'), f(X''), \&\text{c.} \ldots$$

equally converge toward the common limit $f(a)$; and, as they always remain of opposite signs in approaching this limit, it is clear that the quantity $f(a)$, necessarily finite, will not be able to differ from zero. Consequently one will verify the equation

$$(1) \qquad f(x) = 0$$

[15] *propre*, right, suitable.

in giving to the variable x the particular value a comprised between x_0 and X. In other terms,

$$(4) \qquad x = a$$

will be a *root* of equation (1).

[Next, Cauchy presents a series of scholia: the first is on using this proof procedure to find approximate solutions, the second is on what happens when there are multiple possible values for a, the third claims that if $f(x)$ is constantly increasing or decreasing then it has only one root. Cauchy's first corollary is that if there are no roots between x_0 and X then $f(x_0)$ and $f(X)$ have the same sign.]

Corollary 2.nd If in the statement of the 1.st theorem one replaces the function $f(x)$ with $f(x)-b$ (b designating a constant quantity], one will obtain precisely the 4.th theorem of chapter II (§. 2). Under the same hypothesis, following the above indicated method, one will determine numerically the roots of the equation

$$(5) \qquad f(x) = b$$

comprised between x_0 and X.

Appendix B

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