Practice Makes Perfectoid: Pasts, futures, and the historiography of contemporary mathematics

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Abstract:

Comparing my historical account of the early years of Laurent Schwartz's theory of distributions with number theorist Michael Harris's narrative of the early years of Peter Scholze's perfectoid theory, I develop a perspective on change and temporality in mathematics that emphasizes the relationships between concepts, expectations, and communities of practice. Contemporary mathematics, understood as mathematics imbued with temporality, reflects the dynamic relationship between the people, ideas, pasts, and prospects of mathematical knowledge. Studying these historically may offer critical perspectives on the social and political conditions and implications of mathematical research and the communities that practice it.

Key words: expectation, horizon, temporality, confidence, contemporary mathematics, perfectoid theory, distribution theory, concepts, Fields Medal

Prologue (with apologies to Michael Harris)

I first met number theorist Michael Harris when I was an undergraduate student, by way of a number of essays posted to his website. I had no notion at the time that Harris was a number theorist, but the essays made clear that he was a mathematician who was also a clever writer and an insightful reader of French critical theory, all things I too hoped to be. I was particularly inspired by Harris's (1999) remarks on the Science Wars, a cultural conflict of the 1990s that continued to reverberate in the academic contexts where I aspired to study both mathematics and the cultural theories that perhaps (allegedly, controversially) misunderstood or undermined it.

I first introduced myself to Harris some years later, in 2011, at one of the famous coffee breaks at the Institute for Advanced Study in Princeton, New Jersey. I was, by then, a new graduate student in history of science at the neighboring university, part way toward abandoning my ambition to be a mathematician and somewhat less focused on the particular French cultural theories that most excited me as an undergraduate. I had no notion that at that very moment a veritable revolution was unfolding at the Princetitute, that Harris was among the number-theoretic eminences who were in the process of witnessing the spectacular arrival of the perfectoid.

The perfectoid concept was developed and introduced by Peter Scholze, a German mathematician born in the same year I was. Scholze and I began our respective life courses toward Harris in the latter part of the decade in which the precocious Harris "acquired charisma" as a recognized mathematical leader, ratified with a tenured professorship at age 27 (Harris 2015a, ch. 2). Still a graduate student at the time of the perfectoid debut, Scholze completed his doctorate by the end of 2011 and attained a full professorship the subsequent year at age 24.

I first learned of Scholze from Harris two years after our fleeting 2011 introduction, during our first proper conversation in Harris's Paris office. I had come to Harris's mathematics institute to work with the institute's renowned historians of modern mathematics on the planned subject of my doctoral research, the theory of distributions. Introduced by French mathematician Laurent

Schwartz in the latter half of the 1940s, the theory of distributions was an exciting revision to the foundations of mathematical analysis and partial differential equations, earning Schwartz one of the second ever pair of Fields Medals in 1950.

Nearing the end of my Paris research, I asked Harris for an informal meeting. As luck would have it, Harris had just two days prior read a preprint of my report of my sociological research into the material dimensions of mathematical seminars and research practices, from the year prior to my arrival in Princeton (eventually published as Barany and MacKenzie 2014). We had a lot to talk about. I do not remember how the topic came up, but (as can happen in elite mathematics institutes) the conversation turned at one point to future Fields Medalists. I offered a naive guess, which Harris disabused before sharing with great confidence that, sooner or later, the most exciting young mathematician in the world would surely win the coveted prize. The mathematician, whose work Harris was following with great interest and was seeing his students' generation engaging with infectious zeal, was, of course, Scholze.

I caught up with Harris again at a 2014 conference in London (Larvor 2016), by which time I had some thesis results to share (Barany 2016) and Harris could share intimations of his forthcoming monograph, *Mathematics Without Apologies* (Harris 2015a). I was delighted to be invited the next year to review the book for *Mathematical Reviews* (Barany 2015b), my first book review for the venerable bibliographic enterprise whose history had just recently captured my interest as a significant element of mid-twentieth century mathematics (see Barany 2021a). This occasioned a renewed email correspondence with Harris, during which he shared a draft of an essay on the perfectoid concept, which he presented as a "test case for an absent theory."

One thing led to another, and I found myself attempting to revise Harris's draft into a joint essay that would combine Harris's perspective on the emergence of the perfectoid concept with a sociological theory taken from my own work and my interpretation of Harris's recent book. The attempted collaboration fizzled, and Harris's chapter ultimately appeared in a version close to what Harris had generously shared initially (Harris 2017a). For me, the abortive collaboration was a tremendously fruitful chance to rethink my own understanding of what it means to debut an exciting new theory. As I continued to develop my account of the dramatic entry of distributions at mid-century, Harris had primed me to notice the striking convergences in the trajectories of distribution theory and perfectoid theory.

With apologies to the mathematician without apologies, I aim to share here an elaborated account of my theory for his test case, based on Harris's (2017a) narrative of perfectoid theory and my own collaborative account of the corresponding period of Schwartz's career (Barany, Paumier, and Lützen 2017; Barany 2018b). My theory proposes a social-historical perspective on contemporary and unfolding mathematics. I draw from historical theories of the narrative construction of temporality and from sociological theories of the dynamics of confidence and evaluation in mathematical communities. This represents an alternative to the rich and provocative account of the relationship between mathematics and history presented by Blåsjö (2021) in this handbook. The concerns of mathematicians and of critical social historians of mathematics have many significant interfaces, and the perspectives of the latter have much to offer the former.

Making A Promising Theory

Laurent Schwartz was born in 1915 into a world divided by a Great War and brought up in a family of scientists and scholars. He came of age and embarked on advanced mathematical training as his world divided again in the Second World War, excelling in elite settings of French mathematics under mentors connected to some of the most forward-looking and internationally oriented currents of modern mathematical research. As an unestablished researcher in Grenoble, Paris, and Nancy, as the war wound to a close and reconstruction began, he developed the initial ideas that would become his theory of distributions, derived from recent developments in the theory of topological vector spaces.

Schwartz claimed that a radical shift in theoretical perspective could resolve a longstanding foundational concern about the meaning of differential equations involving non-differentiable functions, including functions that had become central in fields such as electrical engineering and quantum physics. He focused initially on potentially intricate technical problems, such as the role of convolution and Fourier duality in his theory, pursued in an intensive French context strongly shaped by the Bourbaki collaboration of pedagogical and foundational radicals. The prestigious 1946 Prix Peccot of the Collège de France gave him a platform in Paris to pursue theoretical questions and build a case and an audience for his theory's significance to modern mathematical analysis.

His theory was based on difficult ideas from recent mathematics, but their implications could be intuited and adopted into mathematical practices in a variety of registers without demanding special expertise in the subjects underpinning Schwartz's arguments. Indeed, for the first half-decade of the theory of distributions, most of the theory's systematic justification and exposition rested in a future textbook that was little more than a promissory note. Most users of the theory could realize its conceptual promise using familiar frames of calculation such as integration by parts, a basic calculus technique whose dramatic reinterpretation in Schwartz's theory demanded very little change to the representational or intuitive dimensions of reckoning about differential equations.

Several influential senior figures of international mathematics encountered Schwartz and his theory in the first years after World War 2, including Norbert Wiener, Marshall Stone, and (most importantly) Harald Bohr. The latter connection was cemented at a 1947 colloquium on harmonic analysis sponsored by the Rockefeller Foundation and Centre National de la Recherche Scientifique as part of an international effort to rebuild outward connections in French science. Bohr was particularly excited by the possibilities he saw in the theory as a bridge between the frontiers of modern pure mathematics and its applications, and supported Schwartz in promoting the theory on those terms. With Bohr's help, Schwartz attracted a wide international interest between 1947 and 1949, including his first international and transoceanic speaking invitations.

Bohr's final intervention, shortly before his death, was to engineer Schwartz's recognition with a Fields Medal at the 1950 International Congress of Mathematicians, decided by a committee Bohr chaired and was not shy about guiding toward his preferred outcome (Barany 2018a). Sensing that his presentation to the 1950 congress might be his last major appearance on the international stage of mathematics, Bohr presented Schwartz's theory as the future of the discipline. He argued that distributions had already transformed basic calculus through the modernizing march of generalization in a way that he predicted would be central to future mathematics curricula.

Things did not go quite as Bohr predicted, but the promise of distributions made Schwartz a sought-after international visitor and interlocutor and helped him secure a stable platform in France on which to build an influential career that did not ultimately depend on the theoretical fruitfulness of early claims about distributions. Mathematicians on multiple continents believed they could build successful careers by engaging, adapting, and teaching the theory of distributions, and they acted on that belief to create an efflorescence of expository texts and proposals for the theory's potential uses and implications. The theory did indeed become foundational in some branches of advanced mathematics, but not necessarily in a way that drove future research programs. Building on his recognition, international travels and connections, and scientific credibility, Schwartz became a public leader in mathematical research, science and education policy, and antiwar and human rights activism (see Journoud 2013, Coeuré 2022).

History Repeating

Peter Scholze was born in 1987 into a world divided by a Cold War and brought up in a family of scientists and scholars. He came of age and embarked on advanced mathematical training in a world still geopolitically defined by war and by global (financial) crisis, albeit not of the sort that surrounded Schwartz, excelling in elite settings of German mathematics under conditions that similarly prepared him for an international orientation to widely recognized mathematical challenges. As a student in Bonn, he developed his perfectoid theory based on recent developments in the field of arithmetic geometry.

Like Schwartz, Scholze debuted his theory with a claim that it resolved a decades-old problem, in his case a special case of the weight-monodromy conjecture. That claim, made without a full demonstration at the Institute for Advanced Study in Princeton in 2011, was followed by a visit and series of six lectures to the Princetitute's Paris counterpart, the Institut des Hautes Études Scientifiques. The first public demonstration took place at these lectures at the IHES, a site of historical and symbolic importance for the problems in algebra, geometry, and number theory for which Scholze's approach appeared to offer a refreshing and spectacular breakthrough. In 2012, as Schwartz had in 1946, Scholze received the Prix Peccot the year after first presenting his theory to the public of mathematical elites.

Some things move quicker in the early 2010s than in the late 1940s. News, videos, and notes of Scholze's presentations were rapidly available across much of the world of arithmetic geometry research. Scholze travelled far and wide, published rapidly, and alone and in collaboration demonstrated a striking series of further uses of his theory. His virtuoso performances of proof and exposition convinced eager audiences, primed to spot the pitfalls of algebraic intuition and ready to be inspired by their circumvention in the resolution of previously stubborn constructions and deductions. As Harris puts it, the cascade of theoretical applications, together with the way the perfectoid concept made sense of relevant antecedent theories, made it clear that this was *the right* (Harris's emphasis) concept to bring together timely combinations of concepts and research agendas from arithmetic geometry. A professorship and associated institutional responsibilities accrued to Scholze much more swiftly than they had for Schwartz.

There were more prizes to win, and Scholze won quite a few of them. He declined one such prize for which Harris had nominated him, for reasons that he did not publicize but about which Harris sympathetically speculated (Harris 2015b; see also Harris 2014). (It will be significant below that the declined prize was called New Horizons.) He did not win a Fields Medal right away, an effect of a major shift in the medal's criteria and perception since 1966 (Barany 2015a). Under the pre-1966 criteria, Scholze would have been unlikely to win at all. His University of Bonn professorial forebear Friedrich Hirzebruch was disqualified from the 1958 medal for having risen too fast in the discipline, and consequently being too well-recognized to qualify under the prevailing older criteria despite his young age (Barany 2018a).

In Harris's telling, Scholze rapidly nucleated a generation shift in his field, supplying new terms, representations, styles, and methods to young researchers who saw his difficult theory as a promising highway toward the vanguard of contemporary research. Graduate students organized study groups to dissect the flood of materials coming from a burgeoning research community. Harris's account recalls the exhilarating sense of motion from supervising a Masters thesis on perfectoid theory begun shortly after Scholze's IHES lectures. By following the theory as it spread and developed in those frenetic months, the student not only rapidly outpaced Harris's own sense of competence in the field in which Scholze intervened, but could chase the theory as far as new results that were unimagined even to Scholze and his collaborators at the time Harris's student had started the project. It was hard to keep up, a difficulty it seems older generations felt more acutely, but those who managed to keep pace with the community of perfectoid researchers were quickly and thrillingly rewarded with access to a fast-moving forefront of mathematical theory.

Berkeley's Mathematical Sciences Research Institute hosted a staggeringly well attended 2014 "Hot Topics" meeting on perfectoid spaces that brought together senior figures who saw their work carried forward in perfectoid research with younger researchers keen to see where the theory could take them. By then, Harris was aware of competent and confident users of perfectoid theory who had not personally closely collaborated with Scholze. This milestone for the reproduction and sustainability of a perfectoid research community corresponded to developments in Scholze's own research trajectory toward *next* concepts and challenges that would go beyond probing the limits of his 2011 formulation.

I spotted Scholze in the queue to enter the opening ceremony of the 2018 International Congress of Mathematicians in Rio de Janeiro. He was conspicuously one of only a very few, especially among the younger attendees, to be wearing something approaching formal attire, as though prepared to be in the spotlight. His was among the least surprising Fields Medal awards in history. It was evident in Scholze's plenary address later in the Congress that the perfectoid concept as such had become just a part of a multifaceted theoretical framework and research program.

Time for Mathematics

Setting Schwartz next to Scholze prepares us to analyze the history of *contemporary mathematics.* By contemporary, I mean mathematics as it is being produced, shared, interrogated, and evaluated. It is mathematics understood to be "now," whenever "now" is. "Now" is a moving target, and its mathematics is therefore always precariously poised between particular pasts and futures, which can alike be mobilized in contemporary mathematical practices. It is thus essential to understand such mathematics as being contemporary in another sense, not just *together in time* (the usual meaning of contemporary) but *imbued with time.* Contemporary mathematics is mathematics pulsing with temporality.

There are two basic ways for mathematics to be *in time*: static or dynamic. Static mathematics exists on a fixed timeline, with ideas and arguments and events each having their place on a linear continuum stretching from past to present to future. The core questions of static mathematics are ontological and epistemological: what things are, what they mean, what they signify or imply, how they relate to each other logically or conceptually or genealogically. These relations can be fixed to identifiable durations: prime numbers in the seventeenth century, epsilon and delta arguments in the 1860s, homotopy type theory pondered over breakfast last week. They can exist outside of duration: the eternal congruences of triangles, the principle of induction, the propositions on page 214 of a textbook. The stuff of static mathematics is separately defined by when it is, where it is, what it means, and so on.

Dynamic mathematics, by contrast, exists at the always-shifting join of specific pasts and futures. There is not a single timeline of dynamic mathematics, because for each person in each place at each moment there may be a different constellation of experiences, understandings, and expectations that shape what mathematical concepts and practices mean (cf. Barany 2017). In the formulation of conceptual historian Reinhart Koselleck (1985), historical time is not a fixed backdrop of human events but a constantly recreated condition generated by the interface of ever-changing experience and ever-retreating horizons of expectation. Rather than what a concept signifies, a dynamic analysis asks (with a subtle but crucial shift in perspective) what its significance is, how it brings together and makes relevant particular experiences and understandings of what the concept relates to, where it comes from, and what it suggests, inspires, or disrupts. The stuff of dynamic mathematics is always on the move, always reconfiguring a multiplicity of pasts and futures (cf. Epple 2011).

Static and dynamic perspectives give two distinct ways to explain the parallel successes of perfectoid and distribution theory. They draw attention to different facets of what matters in the foregoing narratives, with correspondingly different implications for the historical understanding of mathematical practice (cf. Barany 2020). While both kinds of accounts are retrospective, I claim that the retrospection of a static past misrepresents essential features of the contemporary development, adoption, and dispersion of new mathematical ideas identified through a dynamic conception of this history.

Historical and contemporary mathematicians, however, often frame their own conceptions of their past and present in static terms. Customs of narration and of genealogical and conceptual accounting in mathematics can directly obscure the dynamism and instability that is not just intrinsic to contemporary mathematics in the making but is necessary to explain how such mathematics moves and transforms. That is, dynamic practices produce the conditions for the epistemic closure of mathematical understanding in static terms.

Temporalities in Counterpoint

In homage to a narrative device from Harris (2015a), let us consider a series of oppositions posed by a fictional Number Theorist (NT) and Mathematics Historian (MH), categorically not suggested to represent the biographical number theorist and mathematics historian from this chapter's prologue. Specifically, the biographical number theorist Michael Harris deliberately avoided making the kinds of claims in his perfectoid narrative that I shall here attribute to the figure of NT, and in fact often appears (to my reading at least) to take a perspective closer to MH where he does theorize elsewhere. The number theorist will venture static explanations of the early histories of perfectoid and distribution theory, based on a commonsense view of the sociology of mathematics articulated for optimal contrast with the ensuing counterpoint, not on an empirically grounded summary of views of any particular number theorist(s). On the importance of such commonsense endogenous sociologies among mathematicians to the ordinary practice of mathematics, see my chapter with Kremakova in this handbook. The mathematics historian will then reframe NT's static explanations in dynamic terms so that we may consider their different implications for the histories in question.

Our first counterpoint concerns the relationship between mathematical concepts and mathematical communities.

NT: Communities of mathematicians develop mathematical concepts and theories.

MH: Mathematicians use concepts and theories to organize into communities.

Mathematical concepts and theories are, from either perspective, fundamentally communal, developed through exchange, contestation, interpretation, and other communicative activities. For concepts and theories in their infancy, the relationship between communities and concepts runs importantly in both directions: communities change concepts, but concepts are critical loci and resources for changing communities in turn. Concepts and theories can be exciting, inspirational, even *hot,* and these evaluations and designations matter because they transform the communities that engage those concepts and theories. I have in fact elsewhere suggested that, from a social-historical perspective, theories may be *defined* in such terms, as "social systems built around relationships of partial mutual understanding," such that the "theory of distributions" can be defined as a "social system built around relationships of partial mutual understanding of distributions" (Barany 2018b, p. 264). (The significance of the "partial mutual understanding" aspect is a major theme in the rest of this section.) The early formulations of perfectoid spaces, distributional derivatives, and other elements of Schwartz's and Scholze's theories were most historically significant, for reasons we shall explore, for how they could be used to nucleate and expand rapidly growing and highly variegated communities of interested mathematicians with the social, material, technical, and theoretical capacities and motivations to promote the respective theories in a variety of capacities.

This reorientation of the relationship between communities and concepts changes how to interpret the role of those whose names become famous alongside their theories. Both Schwartz and Scholze gained forms of celebrity status with their theories, and their celebrity drew from *and contributed to* the success of their theoretical interventions. Major awards and prizes, likewise, responded to, reshaped, and advanced the mathematicians' theoretical agendas as well as their careers and public images.

> NT: Awards and career milestones follow from, and recognize, important results.
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> MH: Awards and career milestones certify and reinforce the importance of existing results, making them more important for more people, thereby promoting further results and certifications of importance.

The celebrity faces of distribution and perfectoid theory had to convince people in positions to recognize their theories' potential significance, and both succeeded in doing so from early stages in their research and careers. Not only did they convince those with more social and epistemic authority in the discipline of their theories' potential, they were both strikingly successful in establishing that they were the best people to bring their theories to wider and wider mathematics communities. Generous, charismatic, persuasive, and credible, Schwartz and Scholze became recognizable as mathematical celebrities because they were easily legible as potential celebrities.

Prestigious lecture invitations, their respective Prix Peccot and later Fields Medals, and other accolades signaled that these were mathematicians not only worth watching but worth joining. Particularly with Scholze, not just awarded but anticipated recognitions added fuel to the blaze. Conversations like my early one with Harris, speculating about major prizes and future distinctions, lent excitement and momentum, a sense of a theory going somewhere. Both theories and both mathematicians began to receive formal recognitions well in advance of having definitively and widely established the results that would come to be seen as the most significant indications of their importance, and well before there could be said to be a stable consensus about what their interventions really meant for their areas of mathematics. Both mathematicians used the occasions of their honors and associated opportunities to develop their research conversations, introduce further ideas, and project toward the future directions and implications of work in progress.

I have documented how Bohr's enthusiastic and arguably mostly exaggerated impressions of the joint potential of Schwartz and distribution theory motivated Bohr to engineer invitations and recognitions to help that potential be realized (Barany, Paumier, and Lützen 2017). The Fields Medal and Bohr's highly visible characterization of Schwartz and distributions at the 1950 International Congress of Mathematicians, in particular, decisively set a celebrity trajectory that helped Schwartz establish what I suggest is the first mathematical theory from a previously unrecognized mathematician to establish an interconnected and multi-continental research community in its first decade (Barany 2018b). Schwartz charted a path and benefited from nascent institutions that Scholze could traverse and benefit from on a much faster timescale half a century later, helping mathematicians the world over to identify their often-challenging interventions as worth knowing and pursuing.

Public fanfare combined with information infrastructures (preprints, travel, rumor mills, letters, seminars) to build research communities at a pace that far exceeded that of the kind of individual interpersonal interaction that is the usual focus of accounts of collaboration and sense-making in mathematics. To take off or break through requires a conviction of going somewhere. Prizes and careers are simultaneously indications and engines of *going somewhere*, enabling the crucial shifts in perception and scale for theories and names to become big.

> NT: Inspiring leaders of mathematical research communities assume positions of influence and prestige that reflect the leadership they offer.
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> MH: Potential leaders exercise influence and accrue prestige by assuming and representing the roles and expectations of leadership.

The personal and prospective natures of celebrity and leadership in mathematics mean that the visible, recognizable, attributable identities of mathematicians shape their theories and their disciplinary communities. One cannot lead if nobody is there to follow, if whoever is there does not expect one at least potentially to lead. Credibility and leadership are self-reinforcing. Leaders who successfully occupy, justify, and satisfy the optimistic expectations others have of their leadership gain more opportunities and expectations to do the same. Those who do not start with such optimism or who do not fit templates or roles for expectations of leadership, conversely, can do many of the things that would otherwise be associated with innovation and leadership without provoking the loop of expectation and recognition that defines the meteoric careers of others (Rossiter 1993).

From early stages in their careers, both Schwartz and Scholze converted smaller platforms into larger platforms by stepping into and satisfying escalating expectations of mathematical charisma and leadership. Harris (2015a, p. 143) perhaps even understates the case when observing that by age 24 Scholze had become "one of the youngest elder statesmen ever" in his branch of mathematics. As articulate young men, comfortable traveling and confident in settings of academic demonstration and socialization, they launched their careers from a presumption of belonging and potential (see Barany 2021b). Their leadership asked of them qualities and actions they were prepared to exhibit. Both were inarguably exceptional communicators, theorists, and much else, but their success was critically conditioned on, in many other respects, being visibly unexceptional in elite mathematical communities.

Expectations of celebrity and leadership make a big difference to the trajectories of the careers and theories of celebrity leaders, but they also quite profoundly overstate the difference that those celebrity leaders personally make. Like Bohr before him (but thankfully without the same sense of mortal urgency), Harris and a number of comparably positioned established mathematicians followed their somewhat circumstantial encounters with perfectoid theory at its earliest comings-out with later personal and professional activities and commitments to embrace and support the theory and its leading proposer and spokesperson. These established figures, like Bohr and Bohr's generational correspondents, attended lectures, spoke in private and wrote in public, connected students and colleagues, nominated for prizes and otherwise treated perfectoid and distribution theory, and Scholze and Schwartz, as theories and people of which and whom to take note.

In public and in private, Harris has expressed quite strongly that he was marginal to the early trajectory and success of the theory and mathematician he found so exciting. He makes a repeated point in his (2017a) essay and elsewhere (e.g. Harris undated) to disclaim any deep understanding or expertise in perfectoid theory, emphasizing that he has watched and appreciated the rise of perfectoid theory to a significant extent as an outsider. Without wishing to suggest an equivalence in their respective historical roles relative to their theories of interest, I quite suspect Bohr would have felt something similar. Bohr, for his part, was never a recognized mathematical contributor to the theory of distributions as such and was perceived by Schwartz as a somewhat peripheral (though valued) early supporter. Though their roles and situations have many important differences, I think Harris and Bohr each challenge us to reconsider what role such (sometimes self-consciously) theoretically marginal disciplinary elites have in the pivotal formative stages of a fast-rising theory or career.

Bohr's more-than-circumstantial centrality to the history of distributions was a somewhat surprising finding for my collaborators and me based on detailed archival research across many countries and languages (Barany et al. 2017). Without having done the same investigation of perfectoid theory, it is clear enough to me that there are significant differences in precisely how mathematicians in positions of established authority encouraged the latter theory's success. To the extent the comparison is informative here, Harris should be understood as an avatar for the subset of mathematicians who, in parallel to Bohr, recognized and encouraged Scholze's early theory from positions of established professional authority while being peripheral to the core perfectoid research community. In particular, within that community Harris did not appear to play the most Bohr-like role, which might instead be speculatively assigned to Harris's collaborator, the highly decorated number theorist Richard Taylor (an impression Harris has shared with me in more recent correspondence). It was Taylor, for example, who as Distinguished Visiting Professor at the Institute for Advanced Study in 2011 facilitated Scholze's spectacular IAS perfectoid debut. To what extent and to whom further comparisons may apply is a question that may await and surprise future historians.

> NT: Mathematical theories and concepts start with those who best understand them and take hold as more people come to understand them better. Informal and imprecise understandings help motivate and support mathematicians' acquisition of more rigorous understandings.
>
> MH: Mathematical theories and concepts are always simultaneously understood in a variety of ways, and those with middling understanding can be crucial for the theory's early success. Informal understandings, even arguable misunderstandings, often quite adequately replace apparently rigorous understandings, and can be more effective than normatively rigorous accounts at producing and promoting meaningful mathematical knowledge.

The experience of core users and active researchers of a concept and theory only begins to explain its significance and success. Theories and concepts and associated careers necessarily depend on convincing non-experts at different degrees' remove from the active research community. These non-experts must act on partial, provisional, approximate, or proxy understandings to form their evaluations and shape their advocacy.

The unmistakable lesson of the histories of perfectoid and distribution theory is that these non-expert understandings work for the social and intellectual purposes they serve. Among other purposes, such understandings supply a means of organizing and evaluating mathematicians' labor, histories, and values, and these uses are not in general reducible to the technical details that supply the concepts' notional content. It takes an extraordinary amount of work to understand the ideas and methods that have come to be associated with perfectoid spaces, and still more work to adopt those ideas and methods in one's own research. One need not, however, be a perfectoid researcher to be a perfectoid advocate, mentor, sponsor, and (in some settings at least) expositor. Nor need one be a perfectoid expert to appreciate the kinds and extent of difficulty it takes to become such an expert, and to support and reward it accordingly.

Some distance from the active research community can in fact better position an advocate for the social and intellectual roles they play in a theory's success. That distance demands and poises mediating elites to find terms and perspectives that bridge the central ideas and insights of a theory to those who are not (yet or ever) active researchers or experts but who may become experts, interact with them, celebrate or support them, or otherwise be in positions of finding meaning and value in the theory. A small remove from the day-to-day difficulties and frustrations of research with a theory also usefully obscures a sense of its shortcomings, a general phenomenon in the sociology of technical systems that Donald MacKenzie (e.g. 1990, pp. 370-72) has called the certainty trough. Those with the most confidence in a technical system, whether a mathematical research program or a nuclear missile guidance device, are not those with the most or the least expertise. Rather, they tend to be those who know enough to appreciate and be excited by such systems without knowing enough to mitigate that optimistic picture with the quotidian grind of subtle limitations and promises not yet fulfilled.

Even closer to the beating thrum of active research, the multiplicity of meanings matters. Mathematical concepts and theories are always polysemic, always understood in a variety of ways by people attempting to comprehend, use, teach, apply, and reimagine them for mathematical and other purposes. Indeed, many mathematical environments, technologies, and practices appear to be maintained specifically for the purpose of mobilizing unstable and polysemic representations (Barany and MacKenzie 2014). This gives a sociological explanation for the importance of multiple forms and modalities of communication for theories in the making, including venues for localizing and particularizing activities and habits of "reading down" mathematics as a converse to the formalizing work of "writing up" statements and conclusions in stable and circulable forms. I have argued (Barany 2018b) that the proliferation of sometimes mutually conflicting theoretical understandings of the theory of distributions in different research contexts and communities was necessary for its rapid intercontinental trajectory. No one mathematician, Schwartz (or Scholze) included, is ever the comprehensive and exclusive master of everything that makes their theories meaningful, suggestive, useful, valid, or otherwise meritorious. The imagined future uses of a theory, the features that are most important for its adoption and adaptation in growing communities of research and practice, are perhaps most susceptible to and dependent on the constitutive polysemy of dynamic mathematics.

> NT: A theory's antecedents culminate in a new theory, and that culmination is evidence of the theory's promising future. Together, the antecedents and expected future explain (or at

> least support) its successful adoption. Citations acknowledge the dependence of new mathematics on the past.
>
> MH: A theory's antecedents and futures are negotiated (and continually renegotiated) as part of the theory's formation and successful adoption, supporting and shaping what the theory means to those developing and embracing it. Citations identify instrumental pasts that situate new mathematics for the future.

In Harris's and Bohr's respective accountings, there is a striking resemblance in the extent to which Scholze and Schwartz established the promise of their theoretical interventions by successfully mobilizing relevant and suggestive pasts for their theories. In both stories, the most significant antecedents emerge and become canonical as part of processes of explaining and developing a new theory. This frequently happens, and did happen for both of these theories, in dialogue and with the active participation of experts and practitioners in the traditions and theories that become adopted as canonical forerunners and underpinnings of the new theory. While these can be understood in the moment as processes of *recognizing* and *appreciating* those relevant pasts, in a vital dynamic sense the pasts are re-narrated and created as part of the stories and trajectories of the theories for which they become crucial justifications and resources. The pasts link theories to communities of users, suggest conceptual and social affinities, and promise theoretical futures in the form of projections along a train of development. They show current and potential users and appreciators that the theory is (in Harris's term) *the right* one for them, for their goals, values, and interests. Thus canonized, antecedents establish that the theory brings together the pasts that matter to its audiences and shows them a viable way forward.

For Schwartz, and amplified in Bohr's telling, the most significant pasts and antecedents for the theory of distributions could be quite indirectly related to the genealogical configuration of theories and problems from which the theory of distributions first emerged. The theory's early meanings and audiences were thus created in significant part by the addition of temporal and conceptual antecedents that made the new theory meaningful to early interlocutors. In particular, Schwartz grafted distributions onto a conceptual and temporal trajectory of symbolic calculi reaching back through the Dirac calculus of formal derivatives in mathematical physics to the Heaviside calculus in electrical engineering. In this framing, distributions were a culmination of a half-century of symbolic reckoning that motivated the utility and possibility of generalizing the process of differentiation to discontinuous or otherwise non-differentiable functions. Bohr, for his part, enrolled Schwartz in an even longer history of generalization and conceptual closure that included the development and algebraic stabilization of number systems.

These arrangements of pasts for distributions and their associated implicit (and sometimes, especially for Bohr, explicit) futures directly shaped early audiences for distributions and their understandings of the theory. In France, Denmark, and Canada, Schwartz met with crucial early audiences who proved willing to recognize distributions as an ideal solution to a foundational problem about the use of partial differential equations at the interface of pure and applied mathematics. Important advocates following Bohr, such as George Temple in England, domesticated the early framing of the theory as part of a progression of symbolic calculi associated with English theorists. Temple presented his version of Schwartz's theory as an import that reconnected the essential insights and essential utility of distributions to a local tradition of analysis and practical generalization.

The communal and programmatic orientation of the work of establishing canonical antecedents to new theories can be seen in the historical uses of abstracting, bibliography, and citation in modern and recent mathematics (Barany 2021a), as well as in mathematicians' historiographical practice (Paumier and Aubin 2016). Schwartz actively used reviews and citations to establish conceptual and programmatic relationships between his distribution-theoretic work and other bodies of research that may or may not otherwise have had much of anything to do with distributions. The history of citing and acknowledging prior work and of using those citations in ongoing work is thick with the ongoing labor of establishing research communities, reconciling understandings, and demonstrating significance by drawing the past through present projects toward a promising future.

When Scholze introduced perfectoid spaces, he could call upon the skills, techniques, terminologies, and intuitions of the fields at whose crossroads he situated his concept. He could expect researchers in those fields to be prepared to take notice of his intervention, if only he could interest them. These fields gave Scholze a ready means of establishing his work's significance and calling attention to his concept, by providing both a vocabulary for asserting relevance and a stable of open problems that could prove his concept's worth. Importantly, Scholze's early interlocutors saw themselves in the antecedents brought to bear on perfectoid theory, and wrote the history of their own work, priorities, and engagements into the methodological and conceptual trajectories Scholze proposed.

By situating perfectoid spaces both genealogically and analogically in Grothendieck's algebraic geometry and Fontaine's constructions, Scholze supplied a ready basis for affirming their optimality (by analogy to affine and general schemes) and their fruitfulness (by extension of the fruitful genealogy of a tradition in arithmetic geometry). This promise of fruitful applications, based on successfully framing perfectoid spaces as the conceptually appropriate and most practically promising concept for a number of programs of research, helped Scholze recruit (both directly and indirectly) a community of researchers who would begin to fulfill that promise, and thereby reinforce the validity of the initial assessments. Despite the difficulty of perfectoid reckoning, its novelty meant that an enthusiastic young researcher could hope to study up to a meaningful research competency after a relatively short period of intensive study. With the goal of perfectoid geometry in view, such a student could take a short cut (if not a royal road) through only those aspects of the theory's background as seemed most relevant to a desired line of inquiry. Thus, Harris's Masters student could race in the space of a short project from ignorance of the theory almost directly into the community of perfectoid researchers.

Mathematics is, as Harris (2015a) put it, a problematic vocation. That is (in part), mathematics is organized around posing and solving problems (see Barany 2015b). Since fields of study are effectively defined by the programs of active research within them, by frameworks for articulating workable problems and meaningful solutions, the growing programmatic orientation of mathematical researchers toward perfectoid spaces as a unifying concept had the effect of embedding perfectoid geometry as a fundamental framework for a growing body of scholars. Above all, this reflected the self-fulfilling character of being *right,* worthwhile, or fruitful. Scholze's concept had to appear *right* long before its promise for arithmetic geometry began to be realized in great measure and in a wider community of scholars. He set this realization in motion by situating perfectoid spaces in a technical and communal framework that assured his audience of the ability to follow and appreciate his maneuvers and supplied a ready-made basis for establishing their significance. Casting backward to Hodge through Fontaine, Tate, and Grothendieck made it possible to cast forward to a perfectoid-theoretical future.

History and Contemporary Mathematics

I continue to treasure my sporadic exchanges with Harris, who continues to offer a distinctive and invaluable perspective on the contemporary discipline of mathematics. He was one of the first to see a draft of this essay. A recent theme of our conversations on which Harris has also had much to say in public[1] is the political economy of mathematical scholarship and its implications for the social and political responsibilities of mathematicians. Harris is a member of the Azat Miftakhov committee, a group modeled on Schwartz's Committee of Mathematicians to challenge human rights abuses and political imprisonment of mathematicians in our own time, focusing on Miftakhov's political imprisonment (ongoing as of this writing) in Russia. Scholze participated in the group's 2021 conference to raise awareness and show solidarity following Miftakhov's sentencing, an effort to convert some mathematical celebrity into political consciousness.

Harris has made a point of drawing attention to historical analyses, including some of my own, out of a conviction that mathematicians stand to benefit from awareness of the fruits of scholarly research on the social and political history of their field. Part of his exercise of professional and social responsibility is to use his standing as a mathematician and his profile as a cultural critic of mathematics to promote research that helps mathematicians critically interrogate their present and consider the implications of possible futures for the discipline. This kind of engagement and its political and intellectual motivations have a long tradition (see Barany and Kremakova forthcoming).

They are not, however, what Viktor Blåsjö (2021) refers to in this handbook with "Mathematicians used to be highly invested in the study of the history of their own field." Examining the purposes and uses of history "from the mathematician's point of view," Blåsjö advocates for the mathematical value of what he calls "critical internalist" histories of mathematics as a creative fount of mathematical understanding. Blåsjö sees a virtue in mathematicians' endeavor to use the values and methods of conceptual mathematical thinking to make sense of historical mathematics as a form of research experience, one that might build intuition or perspective on the conceptual dimensions of mathematics as a field.

I have suggested here that there are ways of thinking about the power of mathematical ideas that are good for producing mathematical ideas but bad for understanding mathematical communities and the different sorts of power that runs through them. In contrast to Blåsjö's critical internalist historiography, I think this is where a critical social historiography can and should speak to mathematics and mathematicians. An understanding of the inauguration and development of mathematical theories that focuses on static conceptual relations will necessarily miss the most significant dimensions of mathematical creativity and community. These dimensions, of a necessarily dynamic temporality, challenge familiar distinctions between internalist and externalist historiography. Community dynamics have tremendous conceptual consequences, and vice versa. Is hope internal? Is excitement? Is credibility? Is conviction?

From the perspective of the history of contemporary mathematics, of mathematics emphatically temporally in the making, theories and concepts must be understood as both products and tools of social organization, based on the temporalized production and mobilization of relevant pasts and promising horizons. Truth alone does not make a theory successful or even correct (at least in the broader sense I associate with Harris's invocation of being *right*), and is only a small part of its context. Insightfulness, significance, and value are negotiated in communities of practice.

These, in turn, give purchase on the critical potential and importance of the political economies and social contexts of contemporary research. A dynamic temporality of mathematics trains attention on

[1] See esp. ch. 3 of Harris 2015a. See also Harris 2017b and Harris and Jackson 2014.

the facets of mathematical knowledge work where privilege and inequality relate to the central features of mathematics as a system of knowledge and practice, and *vice versa.* Critical social histories can obscure or rationalize as well as identify or challenge the communal-conceptual dynamics of mathematical practice—we might signal the ambivalence of socio-historical analyses by reference to the indecision of history narrated in the *past imperfectoid,* with apologies to Tony Judt's (1992) critical history of the political rationalizations and recriminations of Schwartz's generation. Critical potential is no guarantee of critical purchase, but it is at least a way of interrupting the critical vacuity of static mathematical conceptions of mathematical change.

In his 1543 arithmetic primer, a long distant forerunner to the kind of arithmetic that made Scholze famous, Robert Recorde exhorted that "rules without practice, is but a light knowledge, and practice it is, that maketh men perfect and prompt in all things" (quoted and discussed in Barany 2023). The study of static mathematical rules without dynamic mathematical practice is indeed but a light knowledge. There is, rather, much to be gained from a historiography of contemporary mathematics in the sense sketched here, one in which practice makes not just perfect but also perfectoid.

Cross-References

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