One and a Half Legs to Stand On: The grounds and objects of mathematical proofs.

October 17, 2008

Abstract

Beginning with famous story of Thomas Hobbes's first encounter with the Pythagorean theorem, I ask how the process of verifying a mathematical proof holds together and how such proofs are able to refer to the range of mathematical objects they invoke. Working from the story, as well as some commonplace mathematical examples, I elaborate a simple model for the structure of a mathematical system and introduce the concept of *sesquitextuality* to frame both how this model is engaged by proof-arguments and how such arguments refer to their objects. Sesquitextual systems, I argue, combine concrete workable statements and models with a special sort of allusion in order to allow local and specific claims and examples stand in for global or general ones.

1 The parable of Hobbes and the Theorem

There is an apocryphal story about the great seventeenth century philosopher Thomas Hobbes encountering a manuscript of Euclid's *Elements* opened to the Pythagorean theorem. At first, he could not believe the result. How could every single right triangle conform to such a counter-intuitive equation? Sure that it couldn't be true, he went through the proof, claim by claim, point by point. He examined each cited proposition in turn, the story goes, until he finally reached Euclid's definitions and axioms, which were so fundamental and self-evident as to need no further justification. The theorem was true, after all. This, Hobbes swore, was how philosophy ought to be done.[1]

[1] This story is discussed briefly in Shapin, Steven & Simon Schaffer (1985) *Leviathan and the Air-Pump: Hobbes, Boyle, and the Experimental Life*, pp. 318–319; and Douglas

Today, in lecture after lecture and text after text, the system of mathematics implied in this simple story would seem to be the dominant model of how math is supposed to be done. Rare is the course which does not start with one form or another of first principles, and equally rare is the proof which does not attempt to spell out as precisely as possible the mathematical ground on which it stands. In principle, it would seem, every statement in mathematics ought to submit to such a verification as Hobbes performed, and it is the epistemic system implied by this verification that gives mathematics its mathematical validity.[2]

One could very well frame the philosophy of mathematics in terms of different questions and problems which arise in the parable of Hobbes and the Theorem. For my present purposes, I would like to consider the following two questions:

- First, why did Hobbes believe that the whole system of claims supporting the theorem held together, when it could only have been possible to check the claims, at best, a few at a time?
- And, second, why did Hobbes believe the proof to tell him something about all possible right triangles, and not just the particular ones that may have been in his imagination or in diagrams from the text?

The first question comes from the very human limitation that it is impossible to simultaneously *comprehend* all parts of a proof for all but the most elementary of theorems. One can verify parts in a piecemeal fashion, and verify that parts fit together as claimed, but one cannot, in general, see the whole edifice. The second question is of a similarly practical origin. It would have been impossible to follow the proof, or make sense of it, without some model notion of what it was talking about. In this case, an actual diagrammed triangle may have been involved, or Hobbes could have read the proof with reference to more general notions of what properties a triangle ought to have.[3]

M. Jesseph (1999) "The Decline and Fall of Hobbesian Geometry" in *Studies in the History and Philosophy of Science* 30(3):425–453, p. 426. Both works discuss the broader influence of geometry on Hobbes's philosophical worldview. In particular, see also pp. 100–101 of the former.

[2]Here, I distinguish between *mathematical* validity and other forms of validity such as, for example, validity derived from use value. See Wittgenstein (1972, paperback edition) *Remarks on the Foundations of Mathematics*, Anscombe (trans.), von Wright, Rhees & Anscombe (eds.) for a number of remarks elaborating different forms of mathematical meaning-making.

[3]This latter case is more applicable to most mathematical objects, which, by their very complexity, are completely unrealizable in their entirety. Even such everyday creatures

Put another way, I would like to ask how Hobbes was able to circumscribe the horizons of his mind: How was it that the bit-by-bit functioning of the proof-edifice guaranteed the functioning of the whole proof? How was it that the intelligibility of the theorem for some representative models or images guaranteed the relevance of the theorem for an infinite class of mathematical objects?

In order to begin to answer these questions, I shall introduce the concept of *sesquitextuality*. The term encapsulates a rather simple structural feature of mathematical production which is often papered-over or compartmentalized in analyses of how mathematical proofs work. Mathematics, the above questions would imply, is based upon the successful collaboration of two bodies of knowledge. The first is that which is immediately accessible, apprehensible, and workable. In the parable, this includes individual claims, equations, and diagrams, at the moment at which they are regarded by Hobbes, the proof-checker. The second is that which must be workable and apprehensible *in principle*, but which is neither accessed nor even necessarily accessible *in practice*. For Hobbes, this would include the inaccessible ideal realm of all possible right triangles, as well as cross-sections of the proof and its cited matter too large to be simultaneously and completely comprehended.

Mathematics thus works through a combination of that which is immediately both present and workable and that which is present only through citation or allusion. The latter element has a sort of shadow existence in relation to the former. Unlike the workable features of a proof, the collection of merely cited objects and statements must be taken without the sort of immediate and direct verification normally associated with rigorous mathematical practice. Neither must these statements be taken on blind faith, however. They have a special kind of hypothetical verifiability which exists exclusively in principle. As texts which are constantly *in absentia*, they are a sort of semi-text to the more conventionally verifiable workable texts of the proof. Put together, they form a sesquitextual network of mathematical assertions which combine to produce a valid proof.

2 Human mathematics, ideal mathematics

I intend the concept of sesquitextuality to be a new face for a familiar issue, a face which hopefully allows philosophers of mathematics to view the structure of mathematical proofs in a different light. The familiar issue is a well-

as continuous functions must be accessed through formal characterizations or heuristic models.

tread one. Approximately put, it derives from the fact that mathematics is a human discipline practiced by humans, and the corresponding fact that mathematics itself traffics in ideals.

Generations of logicians, historians, sociologists, mathematicians, and philosophers have made profound achievements with respect to the range of questions emerging from this issue, which I will break into three categories of interaction between humans and ideals.

First, inquiries falling under the broad rubric of logic would seem to treat the way *ideals act on ideals*. Such investigations have explored different logical systems with their associated axioms and rules of inference, and developed sophisticated tools for analyzing each such system and its associated structural possibilities.

Second, treating the way in which *humans act on humans*, are questions about how mathematics is communicated between communities of mathematicians. Wide-ranging studies have investigated the degree to which ambiguities and hidden meanings are built into communication, the degree to which specialized knowledge affects mathematical communication, and the degree to which aspects of arguments can be made explicit.

Third, treating the way in which *humans act on ideals*, are studies of what mathematics means and how that meaning is achieved. Whole bookshelves are filled with scholarly work on the referential character and ontic, epistemic, or other meanings of mathematical statements. Numerous approaches have been applied to inferential methods and processes of reasoning, by which mathematicians negotiate the ideal logical structure of mathematics. Similarly diverse studies concern the way in which mathematical arguments become plausible or credible, and the way in which different mathematical heuristics are developed. I would also include in this category the range of philosophically rich investigations into the process of mathematical discovery and the development of proofs.

While all of these considerations remain important in the critical study of mathematics, and all pertain in significant ways to the concept of sesquitextuality as I am elaborating it, I would like in this paper to focus on a fourth category of human-ideal interaction, namely: *how ideals act on humans*. Thus, the thrust of my above questions about the parable of Hobbes and the Theorem is how the mathematical ideal expressed in the story shaped the production of mathematics described therein. Momentarily leaving aside questions of how humans share mathematics with each other, how mathematical systems work in the abstract, and how humans access mathematical ideals, a study of sesquitextuality may indicate still other ways in which the basic human-ideal conundrum of mathematics manifests itself.

To that end, my analyses below will track two specific aspects of proofmaking concerning the effects of of ideals on the mathematicians who pursue them: workability and citationality. The *workable* elements of a proof are those which, in any given context, are grasped under the finite conditions of even the most ideal variety of human perception and comprehension. They are the parts of ideal mathematics which submit to human analysis. The *citational* parts of a proof are those which invoke arguments which are not immediately present or workable. These are the parts of ideal mathematics which impinge upon human understanding without submitting to analysis.

There is one more lesson to be extracted from the human-ideal problematic before beginning a more thorough development of sesquitextuality. Humans are very particular creatures. They have finite views and finite understandings. Their knowledge and practices are inextricably *local* in character. Ideals, in the other hand, are by their very nature *not* particular. They exist with the utmost of generality, and are thus *global* in character. Mathematics, then, is fundamentally a *local* production creating *global* conclusions. Sesquitextuality will help in framing the way in which the local-global gap created by the interoperation of humans and ideals is navigated in mathematics.

3 Sesquitextual mathematics

Let us now return to my two questions from the opening parable. The first question concerns the structural model of mathematics found in the story, which I will call the *foundation-citation model*. According to this model, mathematics starts with foundations upon which everyone agrees. These foundations could be Euclid's axioms, or Peano's, or Russell's. They could include ideas which are completely self-evident, for example that the whole is greater than any of its parts, and they could also include ideas which are purely conventional, such as $e \cdot x = x \cdot e = x$ if e is a group identity, or persistently controversial, such as the axiom of choice, that it is possible to choose an element from each of an arbitrary collection of sets. They can include fundamental definitions, statements about notation, logical axioms, or rules of inference. The point is that everyone agrees on these foundations from the start.

Foundations in hand, the foundation-citation model then requires that new statements, be they definitions, notations, propositions, or proofs, depend only on statements which have already been established from the foundations. In Hobbes's story, the Pythagorean theorem depended on a number of definitions and propositions, each of which, themselves, depended on further notions, and all of these could be traced back to, and supported entirely by, the axioms at the foundation of Euclid's work. Mathematics becomes a system of statements and citations. Starting with a few statements which are taken to be true, an entire network of true statements is built by adding new statements step-by-step in such a way as to draw only upon statements which have already been established.

One can use the analogy of a brick wall. Some bricks rest firmly on the ground, and can in a certain sense support themselves. These bricks form the foundation. Most bricks, however, do not rest on the ground. Instead, they rest on top of other bricks. An individual brick stays in place because the ones on which it rests stay in place. Each of those stay in place, in turn, because they are either part of the foundation or because they rest on still more bricks which are already in place. In brick wall mathematics, the rules of the game are simple. Start with the foundations, and build up in such a way that each new brick you add has a stable resting place. Put this way, the first question about the parable becomes deceptively simple: if Hobbes can't see the entire wall, how does he know it's still standing?

For real walls, we have a physical explanation. If there were a fault in the wall, then anything which depended in some essential way on this faulty part would come tumbling down. (Indeed, walls are built with a physical-citational redundancy, which mathematics eschews, precisely because physical walls are known to be prone to isolated faults.) In mathematics, however, no such physics come to the rescue. Instead, it is the process of building the wall which lets us know that it stands. Because mathematics is built like a wall, it holds itself up like a wall. The process of building is a local one, concerned just with the next brick and the ones upon which it will immediately rest, whereas the standing of the wall itself is a global consideration.[4]

In mathematics, only a small portion of the justificatory edifice supporting a proof is generally visible, and an even smaller portion can be considered workable. Mathematical proofs are broken up into small claims and arguments for good reason. The ability to synoptically survey an entire claim cannot be dispensed with in its verification, and the reliability of a proof checker declines rapidly as the size of the individual argument to be checked grows, introducing concerns about time, comprehension, and calculation fatigue.[5] For proofs at a sufficiently high level of sophistication, even the

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[4]It is possible to have systems of local accountability implying global functioning without a foundation-based model. Another story involving a prominent philosopher of mathematics illustrates the point. Bertrand Russell was said to have had his cosmology challenged at a public lecture by an old lady in the back of the room, who insisted that the world rests on the back of a giant turtle. Asked what the turtle stood on, she replied 'it's turtles all the way down.' This particular telling was found on Wikipedia, which attributes Stephen Hawking's *Brief History of Time* as its chief popularizer.

[5]See Wittgenstein (1972), *op. cit.*, II: 16–17, and Donald MacKenzie (1999) "Slaying

prospect of checking all of the individual claims needed to completely justify the proof in the way Hobbes did, one at a time, proves practically impossible. There are simply too many claims to be checked.

As mathematics is practiced every day, there thus emerge two global issues for the local work of proof checking. The first, less obvious in our parable, is the sheer volume of what needs to be able to be checked. Mathematicians circumvent this problem by maintaining at least one of the following assertions for each claim which cannot be immediately verified: that it has been verified directly, that it will be verified directly, that it can be verified directly, or that it can certainly be verified in principle. The second issue is the matter of how it all fits together. This concern, for all but the simplest of proofs, is always beyond the scope of local verification. To address it, mathematicians must be confident that the local system they have built just works, and works in a sound and rational way.

The second question from the parable addresses the problem of just what mathematics is talking about. It would be fair to say that nearly all of mathematics is about things which, in themselves, are neither strictly present nor directly workable to the mathematician. The Pythagorean theorem is not, after all, just a theorem about triangles with sides measuring, respectively, 3, 4, and 5 centimeters (leaving aside questions of whether it is possible to construct any such thing with mathematical, rather than scientific, precision). Nor is it a theorem about 3-4-5 right triangles in general. Rather, it is a theorem about all right triangles, of any size and proportion, provided that they are, indeed, right triangles. And it is not only a theorem about right triangles we've seen or are likely to encounter. It is a theorem about all right triangles, possible or impossible, viewable or unviewable, measurable or immeasurable, material or immaterial. To abuse a common philosophical aphorism, one might ask whether a theorem proved about a triangle in a forest, even if nobody is there to see that triangle, still applies.

Mathematics, it would seem, has a special capacity for talking about things it can't access. More importantly, it has the capacity to say something meaningful about inaccessible things in a way which sheds light on what we can, indeed, access. It does so through a process of abstraction, talking about unworkable objects by identifying properties and indicators which are workable. I would venture to say that nobody has seen a continuous function in its full analytic glory, but mathematicians have certainly worked with equations for continuous functions, and the Cauchy criterion allows them to speak about such functions in as much generality as desired. Likewise, one

the Kraken: The Sociohistory of a Mathematical Proof" in *Social Studies of Science* 29(7):7–60, p. 48.

could argue that nobody has seen $1+1=2$ as such, nor $x+x=2x$, but these two tangibly workable material written equations act as perfectly functional surrogates for all that they represent. Proofs involving $x+x=2x$ do not have to establish the proposition for each value of x, much less each thing which could be counted by each value of x, much as proofs involving right triangles need only be performed with reference to a single workable model triangle.

In each case, a surrogate workable object takes the place of the entire range of objects under consideration. These stand-ins come in two types, represented in the above examples. The first type of stand-in is an abstract principle or equation which captures an essential property of the objects in question. Algebraic equations generally serve this purpose. For continuous functions, limit criteria do the job. In Euclid's proof, there were many properties of, for example, angles, triangles, or parallel lines, which needed to be invoked without necessarily referencing a particular model object, even if the conclusions were being applied to one.

The second sort of stand-in involves models, examples, or diagrams. When working through Euclid's proof, one draws or imagines a motley assortment of triangles, squares, and lines. These, themselves, are also not strictly the objects of the proof. Instead, they stand in as workable substitutes for the proof's ideal objects. Geometric manipulations which can be performed in principle on ideal triangles are performed in practice on these drawings.

Of course, formal abstractions and model examples do not necessarily exist in a vacuum. The parallel-line-ness of two lines in a Euclidean figure can be invoked to draw conclusions about the diagram and what it represents on the basis of abstract properties of parallelism. Moreover, models themselves form a type of abstraction of the essential properties of an object. There is a certain sense in which all right triangles *look like* a right triangle in a Euclidean diagram, and this too is a mathematical statement about right triangles in general and in the abstract, although such a statement is not a formal part of the proof.

There is one last ingredient to this system of surrogacy, however. All the while, the ideal triangles or continuous functions are certainly present in the proof, but not wholly present. As represented objects, they are present through the referential signification of their associated models or abstractions. Using these surrogates, the proof is performed *as if* on the objects themselves. This is encoded into the injunctive grammar of the proof itself: 'let ABC be an arbitrary right triangle' or 'take the partial sum of the functions indexed from 1 to n'. Though the proof operates by symbols, it deals in that which the symbols cite—objects which are present only in principle and by citation. Every operation in the proof must, in principle, apply to the objects of the proof, though this is almost never possible in practice.

Present in the proof, then, are two essential elements. First, there are the fully functional models or stand-ins for the objects of the proof, and second are the objects themselves, which are present through citation but nonetheless materially absent from the particular work of the proof. This suggests a first formalization of sesquitextuality as the combination, found in mathematical proofs, of an immediately workable text with a semi-text which must be workable in principle, but which is neither worked nor workable in practice. Both texts must be present for the proof to work and have mathematical meaning.

A second formulation of sesquitextuality comes by classing the scale of these stand-ins and proof objects. While the proof objects are general, or global, in scope, the stand-ins are decidedly local and particular. The proof is checked for the specific example as though it were being checked for the general case, or using the specific abstraction or property as though for the entire range of specimens to which it applies. Thus, sesquitextuality can be characterized as the situation where local accountability serves as the means of accessing global accountability in a proof. The text in this formulation is the local and particular verification with respect to the stand-in, and the semi-text is that extra mathematical feature of the verification which gives it its general scope.

In the foundation-citation model, sesquitextuality appears in both of these formulations. The local accountability of each statement serves to certify the global validity of the proof argument, and the small workable pieces considered in each verification team with the prior but presently inaccessible results and statements they invoke to certify the validity of each claim. Extending the metaphor introduced above, local contact forces between bricks account for the global standing of the wall, and one is able to believe that all of the bricks are in sound order without being able to see the wall all at once.

I am now in a position to begin to frame an answer to the two questions from the opening parable. (Answering them in full, you surely have guessed by now, is far beyond the scope of this work.) In both cases, the sesquitextual structure of mathematics permitted local orders and models to produce global meanings and implications. Moreover, in both questions, limited views of the proof and its objects were given the power to speak as though for all parts and instances through the addition of an allusive gesture which made the whole body of work workable in principle. Hobbes's proof checking and Hobbes's triangles both took sesquitextual shapes.

The concept of sesquitextuality, I claim, offers the possibility of approaching both of these seemingly distinct questions from a single vantage point. An understanding of the operation of sesquitextuality would shed light on the production of mathematical meaning in many corners of a mathematical proof which might otherwise seem irreconcilable. Something about the way mathematics is done—the way proofs are articulated, the way objects are engaged—gives it the capacity to speak to the profound variety of objects to which it speaks and to work in the profoundly robust way in which it works. Mathematics works sesquitextually. It stands on one and a half legs.